

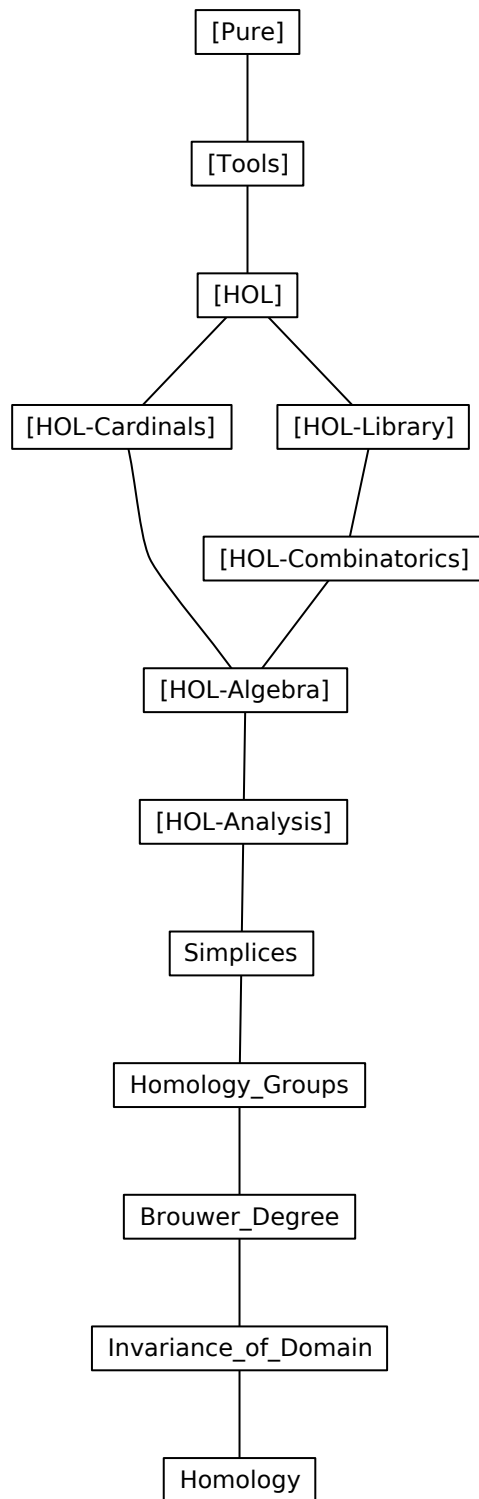
Homology

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0.1 Homology, I: Simplices

theory *Simplices*

imports

HOL-Analysis.Function_Metric

HOL-Analysis.Abstract_Euclidean_Space

HOL-Algebra.Free_Abelian_Groups

begin

- 0.1.1 Standard simplices, all of which are topological subspaces of \mathbb{R}^n .
- 0.1.2 Face map
- 0.1.3 Singular simplices, forcing canonicity outside the intended domain
- 0.1.4 Singular chains
- 0.1.5 Boundary homomorphism for singular chains
- 0.1.6 Factoring out chains in a subtopology for relative homology
- 0.1.7 Relative cycles $Z_p X(S)$ where X is a topology and S a subset
- 0.1.8 Relative boundaries $B_p X(S)$, where X is a topology and S a subset.
- 0.1.9 The (relative) homology relation

- 0.1.10 Show that all boundaries are cycles, the key "chain complex" property.

- 0.1.11 Operations induced by a continuous map g between topological spaces

- 0.1.12 Homology of one-point spaces degenerates except for $p = 0$.

0.1.13 Simplicial chains

0.1.14 The cone construction on simplicial simplices.

0.1.15 Barycentric subdivision of a linear ("simplicial") simplex's image

0.1.16 Singular subdivision

0.1.17 Excision argument that we keep doing singular subdivision

proposition *sufficient_iterated_singular_subdivision_exists:*

assumes $\mathcal{C}: \bigwedge U. U \in \mathcal{C} \implies \text{openin } X \ U$

and $X: \text{topspace } X \subseteq \bigcup \mathcal{C}$

and $p: \text{singular_chain } p \ X \ c$

obtains n **where** $\bigwedge m f. \llbracket n \leq m; f \in \text{Poly_Mapping.keys } ((\text{singular_subdivision } p \ \widehat{\sim} m) \ c) \rrbracket$

$\implies \exists V \in \mathcal{C}. f \ ' (\text{standard_simplex } p) \subseteq V$

0.1.18 Homotopy invariance

theorem *homotopic_imp_homologous_rel_chain_maps:*

assumes $\text{hom}: \text{homotopic_with } (\lambda h. h \ ' T \subseteq V) \ S \ U \ f \ g$ **and** $c: \text{singular_recycle } p \ S \ T \ c$

shows $\text{homologous_rel } p \ U \ V \ (\text{chain_map } p \ f \ c) \ (\text{chain_map } p \ g \ c)$

end

0.2 Homology, II: Homology Groups

theory *Homology_Groups*
imports *Simplices HOL-Algebra.Exact_Sequence*

begin

0.2.1 Homology Groups

0.2.2 Towards the Eilenberg-Steenrod axioms

proposition *homology_homotopy_axiom:*

assumes *homotopic_with* $(\lambda h. h \text{ ' } S \subseteq T) \ X \ Y \ f \ g$

shows $hom_induced \ p \ X \ S \ Y \ T \ f = hom_induced \ p \ X \ S \ Y \ T \ g$

proposition *homology_excision_axiom:*

assumes $X \ closure_of \ U \subseteq X \ interior_of \ T \ T \subseteq S$

shows

$hom_induced \ p \ (subtopology \ X \ (S - U)) \ (T - U) \ (subtopology \ X \ S) \ T \ id$
 $\in iso \ (relative_homology_group \ p \ (subtopology \ X \ (S - U)) \ (T - U))$
 $(relative_homology_group \ p \ (subtopology \ X \ S) \ T)$

0.2.3 Additivity axiom

proposition *iso_cycle_group_sum:*

assumes *disj: pairwise disjnt U* and $UU: \bigcup U = \text{topspace } X$

and subs: $\bigwedge C T. \llbracket \text{compactin } X C; \text{path_connectedin } X C; T \in \mathcal{U}; \neg \text{disjnt } C T \rrbracket \implies C \subseteq T$
shows $(\lambda f. \text{sum}' f \mathcal{U}) \in \text{iso} (\text{sum_group } \mathcal{U} (\lambda T. \text{relcycle_group } p (\text{subtopology } X T) \{\}))$
 $(\text{relcycle_group } p X \{\})$

proposition *homology_additivity_axiom_gen:*

assumes *disj:* pairwise disjnt \mathcal{U} **and** $UU: \bigcup \mathcal{U} = \text{topspace } X$
and subs: $\bigwedge C T. \llbracket \text{compactin } X C; \text{path_connectedin } X C; T \in \mathcal{U}; \neg \text{disjnt } C T \rrbracket \implies C \subseteq T$
shows $(\lambda x. \text{gfinprod} (\text{homology_group } p X)$
 $(\lambda V. \text{hom_induced } p (\text{subtopology } X V) \{\} X \{\} \text{id } (x V)) \mathcal{U})$
 $\in \text{iso} (\text{sum_group } \mathcal{U} (\lambda S. \text{homology_group } p (\text{subtopology } X S))) (\text{homology_group } p X)$
(is $?h \in \text{iso } ?SG ?HG)$

corollary *homology_additivity_axiom:*

assumes *disj:* pairwise disjnt \mathcal{U} **and** $UU: \bigcup \mathcal{U} = \text{topspace } X$
and ope: $\bigwedge v. v \in \mathcal{U} \implies \text{openin } X v$
shows $(\lambda x. \text{gfinprod} (\text{homology_group } p X)$
 $(\lambda v. \text{hom_induced } p (\text{subtopology } X v) \{\} X \{\} \text{id } (x v)) \mathcal{U})$
 $\in \text{iso} (\text{sum_group } \mathcal{U} (\lambda S. \text{homology_group } p (\text{subtopology } X S))) (\text{homology_group } p X)$

0.2.4 Special properties of singular homology

proposition *iso_integer_zeroth_homology_group:*

assumes $X: \text{path_connected_space } X$ **and** $f: \text{singular_simplex } 0 X f$
shows $\text{pow} (\text{homology_group } 0 X) (\text{homologous_rel_set } 0 X \{\} (\text{frag_of } f))$
 $\in \text{iso_integer_group} (\text{homology_group } 0 X) (\text{is } \text{pow } ?H ?q \in \text{iso } _ ?H)$

corollary *isomorphic_integer_zeroth_homology_group:*

assumes $X: \text{path_connected_space } X$ $\text{topspace } X \neq \{\}$
shows $\text{homology_group } 0 X \cong \text{integer_group}$

corollary *homology_coefficients:*

$\text{topspace } X = \{a\} \implies \text{homology_group } 0 X \cong \text{integer_group}$

proposition *zeroth_homology_group:*

$\text{homology_group } 0 X \cong \text{free_Abelian_group} (\text{path_components_of } X)$

0.2.5 More basic properties of homology groups, deduced from the E-S axioms

corollary *mon_hom_induced_section_map:*

assumes *section_map* $X Y f$
shows $(\text{hom_induced } p X \{\} Y \{\} f) \in \text{mon } (\text{homology_group } p X) (\text{homology_group } p Y)$

corollary *epi_hom_induced_retraction_map:*

assumes *retraction_map* $X Y f$
shows $(\text{hom_induced } p X \{\} Y \{\} f) \in \text{epi } (\text{homology_group } p X) (\text{homology_group } p Y)$

0.2.6 Generalize exact homology sequence to triples

proposition *homology_exactness_triple_1:*

assumes $T \subseteq S$
shows $\text{exact_seq } ([\text{relative_homology_group}(p-1) (\text{subtopology } X S) T,$
 $\text{relative_homology_group } p X S,$
 $\text{relative_homology_group } p X T],$
 $[\text{hom_relboundary } p X S T, \text{hom_induced } p X T X S \text{ id}])$
(is exact_seq $([?G1, ?G2, ?G3], [?h1, ?h2]))$

proposition *homology_exactness_triple_2:*

assumes $T \subseteq S$
shows $\text{exact_seq } ([\text{relative_homology_group}(p-1) X T,$
 $\text{relative_homology_group}(p-1) (\text{subtopology } X S) T,$
 $\text{relative_homology_group } p X S],$
 $[\text{hom_induced } (p-1) (\text{subtopology } X S) T X T \text{ id}, \text{hom_relboundary } p X S T])$
(is exact_seq $([?G1, ?G2, ?G3], [?h1, ?h2]))$

proposition *homology_exactness_triple_3:*

assumes $T \subseteq S$

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```
shows exact_seq ([relative_homology_group p X S,
                 relative_homology_group p X T,
                 relative_homology_group p (subtopology X S) T],
                 [hom_induced p X T X S id, hom_induced p (subtopology X S) T
                 X T id])
  (is exact_seq ([?G1, ?G2, ?G3], [?h1, ?h2]))

end
```

0.3 Homology, III: Brouwer Degree

```
theory Brouwer_Degree
  imports Homology_Groups HOL-Algebra.Multiplicative_Group

begin
```

0.3.1 Reduced Homology

0.3.2 More homology properties of deformations, retracts, contractible spaces

```
corollary isomorphic_relative_homology_groups_relativization_contractible:
  assumes contractible_space(subtopology X S) contractible_space(subtopology X
  T) T ⊆ S topspace X ∩ T ≠ {}
  shows relative_homology_group p X T ≅ relative_homology_group p X S
```

```
corollary isomorphic_relative_homology_groups_inclusion_contractible:
  assumes contractible_space X contractible_space(subtopology X S) T ⊆ S topspace
  X ∩ S ≠ {}
```

shows $\text{relative_homology_group } p \text{ (subtopology } X \ S) \ T \cong \text{relative_homology_group } p \ X \ T$

corollary *isomorphic_relative_homology_groups_relboundary_contractible:*

assumes $\text{contractible_space } X \ \text{contractible_space (subtopology } X \ T) \ T \subseteq S \ \text{topspace } X \cap T \neq \{\}$

shows $\text{relative_homology_group } p \ X \ S \cong \text{relative_homology_group } (p - 1) \text{ (subtopology } X \ S) \ T$

0.3.3 Homology groups of spheres

proposition *iso_relative_homology_group_upper_hemisphere:*

$(\text{hom_induced } p \text{ (subtopology (nsphere } n) \ \{x. \ x \ k \geq 0\}) \ \{x. \ x \ k = 0\}) \text{ (nsphere } n) \ \{x. \ x \ k \leq 0\} \ \text{id})$

$\in \text{iso (relative_homology_group } p \text{ (subtopology (nsphere } n) \ \{x. \ x \ k \geq 0\}) \ \{x. \ x \ k = 0\})$

$(\text{relative_homology_group } p \text{ (nsphere } n) \ \{x. \ x \ k \leq 0\}) \text{ (is ?h } \in \text{ iso ?G ?H)}$

corollary *iso_upper_hemisphere_reduced_homology_group:*

$(\text{hom_boundary } (1 + p) \text{ (subtopology (nsphere (Suc } n)) \ \{x. \ x(\text{Suc } n) \geq 0\}) \ \{x. \ x(\text{Suc } n) = 0\})$

$\in \text{iso (relative_homology_group } (1 + p) \text{ (subtopology (nsphere (Suc } n)) \ \{x. \ x(\text{Suc } n) \geq 0\}) \ \{x. \ x(\text{Suc } n) = 0\})$

$(\text{reduced_homology_group } p \text{ (nsphere } n))$

corollary *iso_reduced_homology_group_upper_hemisphere:*

assumes $k \leq n$

shows $\text{hom_induced } p \text{ (nsphere } n) \ \{\} \text{ (nsphere } n) \ \{x. \ x \ k \geq 0\} \ \text{id}$

$\in \text{iso (reduced_homology_group } p \text{ (nsphere } n)) \text{ (relative_homology_group } p \text{ (nsphere } n) \ \{x. \ x \ k \geq 0\})$

0.3.4 Brouwer degree of a Map

corollary *Brouwer_degree2_nonsurjective:*

$\llbracket \text{continuous_map}(\text{nsphere } p) (\text{nsphere } p) f; f \text{ ' } \text{topspace } (\text{nsphere } p) \neq \text{topspace } (\text{nsphere } p) \rrbracket$
 $\implies \text{Brouwer_degree2 } p f = 0$

proposition *Brouwer_degree2_reflection:*

$\text{Brouwer_degree2 } p (\lambda x i. \text{ if } i = 0 \text{ then } -x i \text{ else } x i) = -1$ (**is** $\text{Brouwer_degree2 } p \text{ ?}r = -1$)

end

0.4 Invariance of Domain

theory *Invariance_of_Domain*

imports *Brouwer_Degree HOL-Analysis.Continuous_Extension HOL-Analysis.Homeomorphism*

begin

0.4.1 Degree invariance mod 2 for map between pairs

theorem *Borsuk_odd_mapping_degree_step:*

assumes *cmf*: $\text{continuous_map } (\text{nsphere } n) (\text{nsphere } n) f$
and *f*: $\bigwedge x. x \in \text{topspace}(\text{nsphere } n) \implies (f \circ (\lambda x i. -x i)) x = ((\lambda x i. -x i) \circ f) x$
and *fm*: $f \text{ ' } (\text{topspace}(\text{nsphere}(n - \text{Suc } 0))) \subseteq \text{topspace}(\text{nsphere}(n - \text{Suc } 0))$
shows *even* $(\text{Brouwer_degree2 } n f - \text{Brouwer_degree2 } (n - \text{Suc } 0) f)$

0.4.2 General Jordan-Brouwer separation theorem and invariance of dimension

proposition *relative_homology_group_Euclidean_complement_step:*

assumes *closedin* $(\text{Euclidean_space } n) S$
shows *relative_homology_group* $p (\text{Euclidean_space } n) (\text{topspace}(\text{Euclidean_space } n) - S)$
 $\cong \text{relative_homology_group } (p + k) (\text{Euclidean_space } (n+k)) (\text{topspace}(\text{Euclidean_space } (n+k)) - S)$

proposition *isomorphic_relative_homology_groups_Euclidean_complements:*

assumes *S*: $\text{closedin } (\text{Euclidean_space } n) S$ **and** *T*: $\text{closedin } (\text{Euclidean_space } n) T$
and *hom*: $(\text{subtopology } (\text{Euclidean_space } n) S) \text{ homeomorphic_space } (\text{subtopology } (\text{Euclidean_space } n) T)$
shows *relative_homology_group* $p (\text{Euclidean_space } n) (\text{topspace}(\text{Euclidean_space } n) - S)$

\cong relative_homology_group p (Euclidean_space n) (topspace (Euclidean_space n) - T)

theorem invariance_of_dimension_Euclidean_space:

Euclidean_space m homeomorphic_space Euclidean_space $n \longleftrightarrow m = n$

theorem invariance_of_domain_Euclidean_space:

assumes U : openin (Euclidean_space n) U

and cmf : continuous_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f

and inj_on f U

shows openin (Euclidean_space n) ($f^{-1} U$) (is openin ? E ($f^{-1} U$))

corollary invariance_of_domain_Euclidean_space_embedding_map:

assumes openin (Euclidean_space n) U

and cmf : continuous_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f

and inj_on f U

shows embedding_map (subtopology (Euclidean_space n) U) (Euclidean_space n) f

corollary invariance_of_domain_Euclidean_space_gen:

assumes $n \leq m$ and U : openin (Euclidean_space m) U

and cmf : continuous_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f

and inj_on f U

shows openin (Euclidean_space n) ($f^{-1} U$)

corollary invariance_of_domain_Euclidean_space_embedding_map_gen:

assumes $n \leq m$ and U : openin (Euclidean_space m) U

and cmf : continuous_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f

and inj_on f U

shows embedding_map (subtopology (Euclidean_space m) U) (Euclidean_space n) f

0.4.3 Relating two variants of Euclidean space, one within product topology.

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen_OLC:*

fixes $B :: 'n::\text{euclidean_space set}$

assumes *finite B independent B and orth: pairwise orthogonal B and n: card B = n*

obtains $f\ g$ **where** *homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g*

proposition *homeomorphic_maps_Euclidean_space_euclidean_gen:*

fixes $B :: 'n::\text{euclidean_space set}$

assumes *independent B and orth: pairwise orthogonal B and n: card B = n*

and $1: \bigwedge u. u \in B \implies \text{norm } u = 1$

obtains $f\ g$ **where** *homeomorphic_maps (Euclidean_space n) (top_of_set (span B)) f g*

and $\bigwedge x. x \in \text{topspace (Euclidean_space n)} \implies (\text{norm } (f\ x))^2 = (\sum_{i < n. (x\ i)^2})$

corollary *homeomorphic_maps_Euclidean_space_euclidean:*

obtains $f :: (\text{nat} \implies \text{real}) \implies 'n::\text{euclidean_space}$ **and** g

where *homeomorphic_maps (Euclidean_space (DIM('n))) euclidean f g*

0.4.4 Invariance of dimension and domain

corollary *invariance_of_domain_subspaces:*

fixes $f :: 'a::\text{euclidean_space} \implies 'b::\text{euclidean_space}$

assumes *ope: openin (top_of_set U) S*

and *subspace U subspace V and VU: dim V ≤ dim U*

and *contf: continuous_on S f and fim: f ' S ⊆ V*

and *injf: inj_on f S*

shows *openin (top_of_set V) (f ' S)*

corollary *invariance_of_dimension_subspaces:*

fixes $f :: 'a::\text{euclidean_space} \implies 'b::\text{euclidean_space}$

assumes *ope: openin (top_of_set U) S*

and *subspace U subspace V*

and *contf: continuous_on S f and fim: f ' S ⊆ V*

and *injf: inj_on f S and S ≠ {}*

shows *dim U ≤ dim V*

corollary *invariance_of_domain_affine_sets:*

fixes $f :: 'a::\text{euclidean_space} \implies 'b::\text{euclidean_space}$

assumes *ope: openin (top_of_set U) S*

and *aff: affine U affine V aff_dim V ≤ aff_dim U*

and *contf: continuous_on S f and fim: f ' S ⊆ V*

and *injf: inj_on f S*

shows $\text{openin } (\text{top_of_set } V) (f \text{ ' } S)$

corollary *invariance_of_dimension_affine_sets:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{ope}: \text{openin } (\text{top_of_set } U) S$
and $\text{aff}: \text{affine } U \text{ affine } V$
and $\text{contf}: \text{continuous_on } S f$ **and** $\text{fim}: f \text{ ' } S \subseteq V$
and $\text{injf}: \text{inj_on } f S$ **and** $S \neq \{\}$
shows $\text{aff_dim } U \leq \text{aff_dim } V$

corollary *invariance_of_dimension:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{contf}: \text{continuous_on } S f$ **and** $\text{open } S$
and $\text{injf}: \text{inj_on } f S$ **and** $S \neq \{\}$
shows $\text{DIM}('a) \leq \text{DIM}('b)$

corollary *continuous_injective_image_subspace_dim_le:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{subspace } S \text{ subspace } T$
and $\text{contf}: \text{continuous_on } S f$ **and** $\text{fim}: f \text{ ' } S \subseteq T$
and $\text{injf}: \text{inj_on } f S$
shows $\text{dim } S \leq \text{dim } T$

corollary *invariance_of_domain_homeomorphic:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{open } S \text{ continuous_on } S f \text{ DIM}('b) \leq \text{DIM}('a) \text{ inj_on } f S$
shows $S \text{ homeomorphic } (f \text{ ' } S)$

proposition *homeomorphic_interiors:*

fixes $S :: 'a::\text{euclidean_space set}$ **and** $T :: 'b::\text{euclidean_space set}$
assumes $S \text{ homeomorphic } T \text{ interior } S = \{\} \longleftrightarrow \text{interior } T = \{\}$
shows $(\text{interior } S) \text{ homeomorphic } (\text{interior } T)$

proposition *uniformly_continuous_homeomorphism_UNIV_trivial:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'a$
assumes $\text{contf}: \text{uniformly_continuous_on } S f$ **and** $\text{hom}: \text{homeomorphism } S$
 $\text{UNIV } f g$
shows $S = \text{UNIV}$

proposition *invariance_of_domain_sphere_affine_set_gen:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{contf}: \text{continuous_on } S f$ **and** $\text{injf}: \text{inj_on } f S$ **and** $\text{fim}: f \text{ ' } S \subseteq T$
and $U: \text{bounded } U \text{ convex } U$
and $\text{affine } T$ **and** $\text{affTU}: \text{aff_dim } T < \text{aff_dim } U$
and $\text{ope}: \text{openin } (\text{top_of_set } (\text{rel_frontier } U)) S$
shows $\text{openin } (\text{top_of_set } T) (f \text{ ' } S)$

end

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```
theory Homology  
  imports Invariance_of_Domain  
begin  
  
end
```