

Analysis

May 23, 2024

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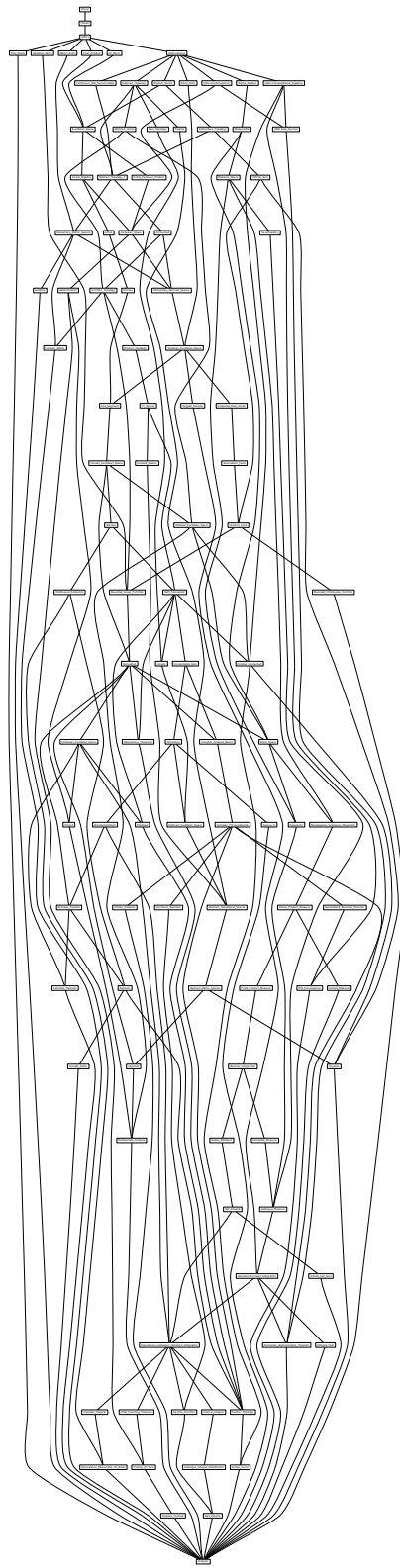
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Chapter 1

Linear Algebra

```
theory L2_Norm
imports Complex_Main
begin
```

1.1 L2 Norm

```
definition L2_set :: ('a ⇒ real) ⇒ 'a set ⇒ real where
L2_set f A = sqrt (∑ i∈A. (f i)2)
```

```
proposition L2_set_triangle_ineq:
  L2_set (λi. f i + g i) A ≤ L2_set f A + L2_set g A
```

```
end
```

1.2 Inner Product Spaces and Gradient Derivative

```
theory Inner_Product
imports Complex_Main
begin
```

1.2.1 Real inner product spaces

```
class real_inner = real_vector + sgn_div_norm + dist_norm + uniformity_dist
+ open_uniformity +
  fixes inner :: 'a ⇒ 'a ⇒ real
  assumes inner_commute: inner x y = inner y x
  and inner_add_left: inner (x + y) z = inner x z + inner y z
  and inner_scaleR_left [simp]: inner (scaleR r x) y = r * (inner x y)
  and inner_ge_zero [simp]: 0 ≤ inner x x
  and inner_eq_zero_iff [simp]: inner x x = 0 ⟷ x = 0
  and norm_eq_sqrt_inner: norm x = sqrt (inner x x)
begin
```

1.2.2 Class instances

```
instantiation real :: real_inner
begin
```

```
instantiation complex :: real_inner
begin
```

1.2.3 Gradient derivative

definition

```
gderiv ::
  ['a::real_inner ⇒ real, 'a, 'a] ⇒ bool
  ((GDERIV (⊂)/ (⊂)/ :> (⊂)) [1000, 1000, 60] 60)
```

where

```
GDERIV f x :> D ⟷ FDERIV f x :> (λh. inner h D)
```

end

1.3 Cartesian Products as Vector Spaces

theory *Product_Vector*

imports

Complex_Main

HOL-Library.Product_Plus

begin

1.3.1 Product is a Module

lemma *scale_prod*: $scale\ x\ (a, b) = (s1\ x\ a, s2\ x\ b)$

sublocale *p*: *module scale*

1.3.2 Product is a Real Vector Space

instantiation *prod* :: $(real_vector, real_vector)$ *real_vector*
begin

proposition *scaleR_Pair* [*simp*]: $scaleR\ r\ (a, b) = (scaleR\ r\ a, scaleR\ r\ b)$

1.3.3 Product is a Metric Space


```

class uniform_topological_monoid_add = topological_monoid_add + uniform_space
+
  assumes uniformly_continuous_add':
    filterlim ( $\lambda((a,b), (c,d)). (a + c, b + d)$ ) uniformity (uniformity  $\times_F$  uniformity)

```

```

class uniform_topological_group_add = topological_group_add + uniform_topological_monoid_add
+
  assumes uniformly_continuous_uinverse': filterlim ( $\lambda(a, b). (-a, -b)$ ) uniformity
uniformity
begin

```

```

instantiation prod :: (metric_space, metric_space) metric_space
begin

```

```

proposition dist_Pair_Pair: dist (a, b) (c, d) = sqrt ((dist a c)2 + (dist b d)2)

```

1.3.4 Product is a Complete Metric Space

```

instance prod :: (complete_space, complete_space) complete_space

```

1.3.5 Product is a Normed Vector Space

```

instantiation prod :: (real_normed_vector, real_normed_vector) real_normed_vector
begin

```

```

proposition norm_Pair: norm (a, b) = sqrt ((norm a)2 + (norm b)2)

```

```

instance prod :: (banach, banach) banach

```

```

proposition has_derivative_Pair [derivative_intros]:

```

```

  assumes f: (f has_derivative f') (at x within s)

```

```

    and g: (g has_derivative g') (at x within s)

```

```

  shows (( $\lambda x. (f x, g x)$ ) has_derivative ( $\lambda h. (f' h, g' h)$ )) (at x within s)

```

1.3.6 Product is Finite Dimensional

```

proposition dim_Times:

```

```

  assumes vs1.subspace S vs2.subspace T

```

```

  shows p.dim(S  $\times$  T) = vs1.dim S + vs2.dim T

```

```

end

```

1.4 Finite-Dimensional Inner Product Spaces

```

theory Euclidean_Space
imports
  L2_Norm
  Inner_Product
  Product_Vector
begin

```

1.4.1 Type class of Euclidean spaces

```

class euclidean_space = real_inner +
  fixes Basis :: 'a set
  assumes nonempty_Basis [simp]: Basis ≠ {}
  assumes finite_Basis [simp]: finite Basis
  assumes inner_Basis:
     $[[u \in \text{Basis}; v \in \text{Basis}] \implies \text{inner } u \ v = (\text{if } u = v \text{ then } 1 \text{ else } 0)$ 
  assumes euclidean_all_zero_iff:
     $(\forall u \in \text{Basis}. \text{inner } x \ u = 0) \longleftrightarrow (x = 0)$ 

```

1.4.2 Class instances

```

instantiation real :: euclidean_space
begin
instantiation complex :: euclidean_space
begin
instantiation prod :: (real_inner, real_inner) real_inner
begin

instantiation prod :: (euclidean_space, euclidean_space) euclidean_space
begin

```

1.4.3 Locale instances

```

end

```

1.5 Elementary Linear Algebra on Euclidean Spaces

```

theory Linear_Algebra
imports
  Euclidean_Space
  HOL-Library.Infinite_Set
begin

```

1.5.1 Substandard Basis

1.5.2 Orthogonality

definition (in *real_inner*) *orthogonal* $x\ y \longleftrightarrow x \cdot y = 0$

1.5.3 Orthogonality of a transformation

definition *orthogonal_transformation* $f \longleftrightarrow \text{linear } f \wedge (\forall v\ w. f\ v \cdot f\ w = v \cdot w)$

1.5.4 Bilinear functions

definition

bilinear :: ('a::real_vector \Rightarrow 'b::real_vector \Rightarrow 'c::real_vector) \Rightarrow bool **where**
bilinear $f \longleftrightarrow (\forall x. \text{linear } (\lambda y. f\ x\ y)) \wedge (\forall y. \text{linear } (\lambda x. f\ x\ y))$

1.5.5 Adjoints

definition *adjoint* :: (('a::real_inner) \Rightarrow ('b::real_inner)) \Rightarrow 'b \Rightarrow 'a **where**
adjoint $f = (\text{SOME } f'. \forall x\ y. f\ x \cdot y = x \cdot f'\ y)$

1.5.6 Infinity norm

definition *infnorm* ($x::'a::\text{euclidean_space}$) = *Sup* $\{|x \cdot b| \mid b. b \in \text{Basis}\}$

1.5.7 Collinearity

definition *collinear* :: 'a::real_vector set \Rightarrow bool
where *collinear* $S \longleftrightarrow (\exists u. \forall x \in S. \forall y \in S. \exists c. x - y = c *_R u)$

1.5.8 Properties of special hyperplanes

proposition *dim_hyperplane*:

fixes $a :: 'a::\text{euclidean_space}$

assumes $a \neq 0$

shows $\text{dim } \{x. a \cdot x = 0\} = \text{DIM}(a) - 1$

1.5.9 Orthogonal bases and Gram-Schmidt process

proposition *Gram_Schmidt_step*:

fixes $S :: 'a::\text{euclidean_space}$ set

assumes S : pairwise orthogonal S **and** $x: x \in \text{span } S$

shows orthogonal x ($a - (\sum_{b \in S}. (b \cdot a / (b \cdot b)) *_{\mathbb{R}} b)$)

proposition *orthogonal_extension*:

fixes $S :: 'a::euclidean_space$ set

assumes S : pairwise orthogonal S

obtains U where pairwise orthogonal $(S \cup U)$ $\text{span } (S \cup U) = \text{span } (S \cup T)$

1.5.10 Decomposing a vector into parts in orthogonal subspaces

proposition *orthonormal_basis_subspace*:

fixes $S :: 'a :: euclidean_space$ set

assumes subspace S

obtains B where $B \subseteq S$ pairwise orthogonal B

and $\bigwedge x. x \in B \implies \text{norm } x = 1$

and independent B $\text{card } B = \text{dim } S$ $\text{span } B = S$

proposition *dim_orthogonal_sum*:

fixes $A :: 'a::euclidean_space$ set

assumes $\bigwedge x y. \llbracket x \in A; y \in B \rrbracket \implies x \cdot y = 0$

shows $\text{dim}(A \cup B) = \text{dim } A + \text{dim } B$

1.5.11 Linear functions are (uniformly) continuous on any set

end

1.6 Affine Sets

theory *Affine*

imports *Linear_Algebra*

begin

1.6.1 Affine set and affine hull

definition *affine* :: $'a::real_vector$ set \implies bool

where $\text{affine } S \iff (\forall x \in S. \forall y \in S. \forall u v. u + v = 1 \longrightarrow u *_{\mathbb{R}} x + v *_{\mathbb{R}} y \in S)$

1.6.2 Affine Dependence

definition *affine_dependent* :: 'a::real_vector set \Rightarrow bool
 where *affine_dependent* S \longleftrightarrow ($\exists x \in S. x \in \text{affine hull } (S - \{x\})$)

proposition *affine_dependent_explicit*:
affine_dependent p \longleftrightarrow
 ($\exists S U. \text{finite } S \wedge S \subseteq p \wedge \text{sum } U S = 0 \wedge (\exists v \in S. U v \neq 0) \wedge \text{sum } (\lambda v. U v *_{\mathbb{R}} v) S = 0$)

proposition *extend_to_affine_basis*:
 fixes S V :: 'n::real_vector set
 assumes $\neg \text{affine_dependent } S S \subseteq V$
 obtains T where $\neg \text{affine_dependent } T S \subseteq T T \subseteq V \text{ affine hull } T = \text{affine hull } V$

1.6.3 Affine Dimension of a Set

definition *aff_dim* :: ('a::euclidean_space) set \Rightarrow int
 where *aff_dim* V =
 (SOME d :: int.
 $\exists B. \text{affine hull } B = \text{affine hull } V \wedge \neg \text{affine_dependent } B \wedge \text{of_nat } (\text{card } B) = d + 1$)

end

1.7 Convex Sets and Functions

theory Convex
imports
 Affine HOL-Library.Set_Algebras HOL-Library.FuncSet
begin

1.7.1 Convex Sets

definition *convex* :: 'a::real_vector set \Rightarrow bool
 where *convex* s \longleftrightarrow ($\forall x \in s. \forall y \in s. \forall u \geq 0. \forall v \geq 0. u + v = 1 \longrightarrow u *_{\mathbb{R}} x + v *_{\mathbb{R}} y \in s$)

1.7.2 Convex Functions on a Set

definition *convex_on* :: 'a::real_vector set \Rightarrow ('a \Rightarrow real) \Rightarrow bool
 where *convex_on* S f \longleftrightarrow *convex* S \wedge
 ($\forall x \in S. \forall y \in S. \forall u \geq 0. \forall v \geq 0. u + v = 1 \longrightarrow f (u *_{\mathbb{R}} x + v *_{\mathbb{R}} y) \leq u * f x + v * f y$)

definition *concave_on* :: 'a::real_vector set \Rightarrow ('a \Rightarrow real) \Rightarrow bool
where *concave_on* S f \equiv *convex_on* S ($\lambda x. - f x$)

1.7.3 Some inequalities: Applications of convexity

1.7.4 Misc related lemmas

1.7.5 Cones

definition *cone* :: 'a::real_vector set \Rightarrow bool
where *cone* s \longleftrightarrow ($\forall x \in s. \forall c \geq 0. c *_{\mathbb{R}} x \in s$)

proposition *cone_hull_expl*: *cone hull* S = {c *_R x | c x. c \geq 0 \wedge x \in S}
(is ?lhs = ?rhs)

1.7.6 Convex hull

proposition *convex_hull_indexed*:

fixes S :: 'a::real_vector set

shows *convex hull* S =

{y. $\exists k u x. (\forall i \in \{1..k\}. 0 \leq u i \wedge x i \in S) \wedge$
 $(\text{sum } u \{1..k\} = 1) \wedge (\sum_{i=1..k} u i *_{\mathbb{R}} x i) = y$ }

(is ?xyz = ?hull)

1.7.7 Caratheodory's theorem

theorem *caratheodory*:

convex hull p =

{x::'a::euclidean_space. $\exists S. \text{finite } S \wedge S \subseteq p \wedge \text{card } S \leq \text{DIM('a)} + 1 \wedge x \in$
convex hull S}

1.8 Conic sets and conic hull

1.9 Convex cones and corresponding hulls

1.9.1 Radon's theorem

theorem *Radon*:

assumes *affine_dependent* c

obtains M P **where** M \subseteq c P \subseteq c M \cap P = {} (*convex hull* M) \cap (*convex hull* P) \neq {}

1.9.2 Helly's theorem

theorem *Helly*:

fixes $\mathcal{F} :: 'a::\text{euclidean_space}$ *set set*
assumes $\text{card } \mathcal{F} \geq \text{DIM}('a) + 1 \ \forall s \in \mathcal{F}. \text{convex } s$
and $\bigwedge t. [\![t \subseteq \mathcal{F}; \text{card } t = \text{DIM}('a) + 1]\!] \implies \bigcap t \neq \{\}$
shows $\bigcap \mathcal{F} \neq \{\}$

1.9.3 Epigraphs of convex functions

definition *epigraph* S ($f :: _ \Rightarrow \text{real}$) = $\{xy. \text{fst } xy \in S \wedge f(\text{fst } xy) \leq \text{snd } xy\}$

end

1.10 Definition of Finite Cartesian Product Type

theory *Finite_Cartesian_Product*

imports

Euclidean_Space
L2_Norm
HOL-Library.Numeral_Type
HOL-Library.Countable_Set
HOL-Library.FuncSet

begin

1.10.1 Cardinality of vectors

proposition *CARD_vec* [*simp*]:

$\text{CARD}('a \wedge 'b) = \text{CARD}('a) \wedge \text{CARD}('b)$

instantiation *vec* :: (*zero, finite*) *zero*

begin

instantiation *vec* :: (*plus, finite*) *plus*

begin

instantiation *vec* :: (*minus, finite*) *minus*

begin

instantiation *vec* :: (*uminus, finite*) *uminus*

begin

instantiation *vec* :: (*times, finite*) *times*

begin

instantiation *vec* :: (*one, finite*) *one*

begin

instantiation *vec* :: (*ord*, *finite*) *ord*
begin

1.10.2 Real vector space

definition *scaleR* ≡ (λ *r x*. (χ *i*. *scaleR* *r* (*x*\$*i*)))

1.10.3 Topological space

definition [*code del*]:
open (*S* :: ('*a* ^ '*b*) *set*) ↔
 (∀ *x* ∈ *S*. ∃ *A*. (∀ *i*. *open* (*A* *i*) ∧ *x*\$*i* ∈ *A* *i*) ∧
 (∀ *y*. (∀ *i*. *y*\$*i* ∈ *A* *i*) → *y* ∈ *S*))

1.10.4 Metric space

definition
dist *x y* = *L2_set* (λ *i*. *dist* (*x*\$*i*) (*y*\$*i*)) *UNIV*

definition [*code del*]:
 (*uniformity* :: (('*a* ^ '*b*::_) × ('*a* ^ '*b*::_)) *filter*) =
 (INF *e* ∈ {0 <..}. *principal* {(*x*, *y*). *dist* *x y* < *e*})

proposition *dist_vec_nth_le*: *dist* (*x* \$ *i*) (*y* \$ *i*) ≤ *dist* *x y*

1.10.5 Normed vector space

definition *norm* *x* = *L2_set* (λ *i*. *norm* (*x*\$*i*)) *UNIV*

definition *sgn* (*x*::'*a* ^ '*b*) = *scaleR* (*inverse* (*norm* *x*)) *x*

1.10.6 Inner product space

definition *inner* *x y* = *sum* (λ *i*. *inner* (*x*\$*i*) (*y*\$*i*)) *UNIV*

1.10.7 Euclidean space

definition $axis\ k\ x = (\chi\ i.\ \text{if } i = k \text{ then } x \text{ else } 0)$

definition $Basis = (\bigcup i.\ \bigcup u \in Basis.\ \{axis\ i\ u\})$

proposition $DIM_cart\ [simp]: DIM('a \wedge 'b) = CARD('b) * DIM('a)$

1.10.8 Matrix operations

definition $map_matrix::('a \Rightarrow 'b) \Rightarrow (('a, 'i::finite)vec, 'j::finite)vec \Rightarrow (('b, 'i)vec, 'j)vec$ **where**
 $map_matrix\ f\ x = (\chi\ i\ j.\ f\ (x\ \$\ i\ \$\ j))$

definition $matrix_matrix_mult :: ('a::semiring_1) \wedge^n \wedge^m \Rightarrow 'a \wedge^p \wedge^n \Rightarrow 'a \wedge^p \wedge^m$
(infixl ** 70)
where $m ** m' == (\chi\ i\ j.\ sum\ (\lambda k.\ ((m\ \$\ i)\ \$\ k) * ((m'\ \$\ k)\ \$\ j))\ (UNIV :: 'n\ set)) :: 'a \wedge^p \wedge^m$

definition $matrix_vector_mult :: ('a::semiring_1) \wedge^n \wedge^m \Rightarrow 'a \wedge^n \Rightarrow 'a \wedge^m$
(infixl *v 70)
where $m *v\ x \equiv (\chi\ i.\ sum\ (\lambda j.\ ((m\ \$\ i)\ \$\ j) * (x\ \$\ j))\ (UNIV :: 'n\ set)) :: 'a \wedge^m$

definition $vector_matrix_mult :: 'a \wedge^m \Rightarrow ('a::semiring_1) \wedge^n \wedge^m \Rightarrow 'a \wedge^n$
(infixl v* 70)
where $v *v\ m == (\chi\ j.\ sum\ (\lambda i.\ ((m\ \$\ i)\ \$\ j) * (v\ \$\ i))\ (UNIV :: 'm\ set)) :: 'a \wedge^n$

proposition $matrix_mul_assoc: A ** (B ** C) = (A ** B) ** C$

proposition $matrix_vector_mul_assoc: A *v (B *v x) = (A ** B) *v x$

proposition $scalar_matrix_assoc:$
fixes $A :: ('a::real_algebra_1) \wedge^m \wedge^n$
shows $k *_R (A ** B) = (k *_R A) ** B$

proposition $matrix_scalar_ac:$
fixes $A :: ('a::real_algebra_1) \wedge^m \wedge^n$
shows $A ** (k *_R B) = k *_R A ** B$

definition $matrix :: ('a::\{plus,times,one,zero\}) \wedge^m \Rightarrow 'a \wedge^n \Rightarrow 'a \wedge^m \wedge^n$
where $matrix\ f = (\chi\ i\ j.\ (f(axis\ j\ 1))\ \$\ i)$

1.10.9 Inverse matrices (not necessarily square)

definition
 $invertible(A::'a::semiring_1 \wedge^n \wedge^m) \longleftrightarrow (\exists A'::'a \wedge^m \wedge^n.\ A ** A' = mat\ 1 \wedge A' ** A = mat\ 1)$

definition

matrix_inv($A :: 'a :: \text{semiring_1}^n{}^m$) =
 (SOME $A' :: 'a^m{}^n$. $A ** A' = \text{mat } 1 \wedge A' ** A = \text{mat } 1$)

proposition *scalar_invertible_iff*:

fixes $A :: ('a :: \text{real_algebra_1})^m{}^n$
assumes $k \neq 0$ **and** *invertible* A
shows *invertible* ($k *_R A$) $\longleftrightarrow k \neq 0 \wedge \text{invertible } A$

proposition *vector_scaleR_matrix_ac*:

fixes $k :: \text{real}$ **and** $x :: \text{real}^n$ **and** $A :: \text{real}^m{}^n$
shows $x v* (k *_R A) = k *_R (x v* A)$

end

1.11 Linear Algebra on Finite Cartesian Products

theory *Cartesian_Space***imports***HOL-Combinatorics.Transposition**Finite_Cartesian_Product**Linear_Algebra***begin**

1.11.1 Some interesting theorems and interpretations

1.11.2 Rank of a matrix

definition *rank* :: $'a :: \text{field}^n{}^m \Rightarrow \text{nat}$ **where** *row_rank_def_gen*: $\text{rank } A \equiv \text{vec.dim}(\text{rows } A)$

1.11.3 Orthogonality of a matrix

definition *orthogonal_matrix* ($Q :: 'a :: \text{semiring_1}^n{}^n$) \longleftrightarrow *transpose* $Q ** Q = \text{mat } 1 \wedge Q ** \text{transpose } Q = \text{mat } 1$ **proposition** *orthogonal_matrix_mul*:

fixes $A :: \text{real}^n{}^n$
assumes *orthogonal_matrix* A *orthogonal_matrix* B
shows *orthogonal_matrix*($A ** B$)

proposition *orthogonal_transformation_matrix*:

fixes $f :: \text{real}^n \Rightarrow \text{real}^n$
shows $\text{orthogonal_transformation } f \longleftrightarrow \text{linear } f \wedge \text{orthogonal_matrix}(\text{matrix } f)$
(is $?lhs \longleftrightarrow ?rhs)$

1.11.4 Finding an Orthogonal Matrix

proposition $\text{orthogonal_matrix_exists_basis}$:
fixes $a :: \text{real}^n$
assumes $\text{norm } a = 1$
obtains A **where** $\text{orthogonal_matrix } A \wedge A * v (\text{axis } k \ 1) = a$

proposition $\text{orthogonal_transformation_exists}$:
fixes $a \ b :: \text{real}^n$
assumes $\text{norm } a = \text{norm } b$
obtains f **where** $\text{orthogonal_transformation } f \wedge f a = b$

1.11.5 Scaling and isometry

proposition scaling_linear :
fixes $f :: 'a::\text{real_inner} \Rightarrow 'a::\text{real_inner}$
assumes $f0: f \ 0 = 0$
and $fd: \forall x \ y. \text{dist } (f \ x) \ (f \ y) = c * \text{dist } x \ y$
shows $\text{linear } f$

proposition $\text{orthogonal_transformation_isometry}$:
 $\text{orthogonal_transformation } f \longleftrightarrow f(0::'a::\text{real_inner}) = (0::'a) \wedge (\forall x \ y. \text{dist}(f \ x) \ (f \ y) = \text{dist } x \ y)$

1.11.6 Induction on matrix row operations

end

1.12 Traces and Determinants of Square Matrices

theory Determinants
imports
 $\text{HOL-Combinatorics.Permutations}$
 Cartesian_Space
begin

1.12.1 Trace

definition $\text{trace} :: 'a::\text{semiring_1}^n \Rightarrow 'a$
where $\text{trace } A = \text{sum } (\lambda i. ((A\$i)\$i)) \ (\text{UNIV}::'n \ \text{set})$

Definition of determinant

definition $det :: 'a :: comm_ring \Rightarrow 'a$ **where**

$det A =$
 $sum (\lambda p. of_int (sign p) * prod (\lambda i. A\$i\$p i) (UNIV :: 'n set))$
 $\{p. p \text{ permutes } (UNIV :: 'n set)\}$

proposition $det_diagonal$:

fixes $A :: 'a :: comm_ring$
assumes $ld: \bigwedge i j. i \neq j \implies A\$i\$j = 0$
shows $det A = prod (\lambda i. A\$i\$i) (UNIV :: 'n set)$

proposition $det_matrix_scaleR [simp]: det (matrix (((*_R) r)) :: real^{n \times n}) = r^{CARD('n)}$

proposition det_mul :

fixes $A B :: 'a :: comm_ring$
shows $det (A ** B) = det A * det B$

1.12.2 Relation to invertibility

proposition $invertible_det_nz$:

fixes $A :: 'a :: field^{n \times n}$
shows $invertible A \iff det A \neq 0$

Invertibility of matrices and corresponding linear functions

1.12.3 Cramer's rule

proposition $cramer_lemma$:

fixes $A :: 'a :: field^{n \times n}$
shows $det((\chi i j. if j = k then (A * v x)\$i else A\$i\$j) :: 'a :: field^{n \times n}) = x\$k * det A$

proposition $cramer$:

fixes $A :: 'a :: field^{n \times n}$
assumes $d0: det A \neq 0$
shows $A * v x = b \iff x = (\chi k. det(\chi i j. if j=k then b\$i else A\$i\$j) / det A)$

proposition $det_orthogonal_matrix$:

fixes $Q :: 'a :: linordered_idom^{n \times n}$
assumes $oQ: orthogonal_matrix Q$
shows $det Q = 1 \vee det Q = -1$

proposition $orthogonal_transformation_det [simp]$:

fixes $f :: real^{n \times n} \Rightarrow real^{n \times n}$

shows $orthogonal_transformation\ f \implies |det\ (matrix\ f)| = 1$

1.12.4 Rotation, reflection, rotoinversion

definition $rotation_matrix\ Q \longleftrightarrow orthogonal_matrix\ Q \wedge det\ Q = 1$

definition $rotoinversion_matrix\ Q \longleftrightarrow orthogonal_matrix\ Q \wedge det\ Q = -1$

end

1.13 Operators involving abstract topology

theory *Abstract_Topology*

imports

Complex_Main

HOL-Library.Set_Idioms

HOL-Library.FuncSet

begin

1.13.1 General notion of a topology as a value

definition $istopology :: ('a\ set \Rightarrow bool) \Rightarrow bool$ **where**

$istopology\ L \equiv (\forall S\ T. L\ S \longrightarrow L\ T \longrightarrow L\ (S \cap T)) \wedge (\forall \mathcal{K}. (\forall K \in \mathcal{K}. L\ K) \longrightarrow L\ (\bigcup \mathcal{K}))$

typedef $'a\ topology = \{L :: ('a\ set) \Rightarrow bool. istopology\ L\}$

morphisms *openin topology*

proposition *openin_clauses:*

fixes $U :: 'a\ topology$

shows

$openin\ U\ \{\}$

$\bigwedge S\ T. openin\ U\ S \implies openin\ U\ T \implies openin\ U\ (S \cap T)$

$\bigwedge K. (\forall S \in K. openin\ U\ S) \implies openin\ U\ (\bigcup K)$

definition $closedin :: 'a\ topology \Rightarrow 'a\ set \Rightarrow bool$ **where**

$closedin\ U\ S \longleftrightarrow S \subseteq topspace\ U \wedge openin\ U\ (topspace\ U - S)$

1.13.2 The discrete topology

1.13.3 Subspace topology

definition $subtopology :: 'a\ topology \Rightarrow 'a\ set \Rightarrow 'a\ topology$

where $subtopology\ U\ V = topology\ (\lambda T. \exists S. T = S \cap V \wedge openin\ U\ S)$

1.13.4 The canonical topology from the underlying type class

abbreviation $euclidean :: 'a::topological_space\ topology$

where $euclidean \equiv topology\ open$

1.13.5 Basic "localization" results are handy for connectedness.

1.13.6 Derived set (set of limit points)

1.13.7 Closure with respect to a topological space

1.13.8 Frontier with respect to topological space

1.13.9 Locally finite collections

1.13.10 Continuous maps

lemma *continuous_map_alt:*

$$\begin{aligned} & \text{continuous_map } T1 \ T2 \ f \\ & = ((\forall U. \text{openin } T2 \ U \longrightarrow \text{openin } T1 \ (f^{-1} U \cap \text{topspace } T1)) \wedge f \in \text{topspace } \\ & T1 \rightarrow \text{topspace } T2) \end{aligned}$$

1.13.11 Open and closed maps (not a priori assumed continuous)

1.13.12 Quotient maps

1.13.13 Separated Sets

1.13.14 Homeomorphisms

1.13.15 Relation of homeomorphism between topological spaces

1.13.16 Connected topological spaces

1.13.17 Compact sets

proposition *compact_space_fip:*

$compact_space\ X \longleftrightarrow$
 $(\forall \mathcal{U}. (\forall C \in \mathcal{U}. closedin\ X\ C) \wedge (\forall \mathcal{F}. finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow \bigcap \mathcal{F} \neq \{\}) \longrightarrow$
 $\bigcap \mathcal{U} \neq \{\})$
 $(is_ = ?rhs)$

corollary *compactin_fip*:

$compactin\ X\ S \longleftrightarrow$
 $S \subseteq topspace\ X \wedge$
 $(\forall \mathcal{U}. (\forall C \in \mathcal{U}. closedin\ X\ C) \wedge (\forall \mathcal{F}. finite\ \mathcal{F} \wedge \mathcal{F} \subseteq \mathcal{U} \longrightarrow S \cap \bigcap \mathcal{F} \neq \{\}) \longrightarrow$
 $S \cap \bigcap \mathcal{U} \neq \{\})$

corollary *compact_space_imp_nest*:

fixes $C :: nat \Rightarrow 'a\ set$
assumes $compact_space\ X$ **and** $clo: \bigwedge n. closedin\ X\ (C\ n)$
and $ne: \bigwedge n. C\ n \neq \{\}$ **and** $dec: decseq\ C$
shows $(\bigcap n. C\ n) \neq \{\}$

1.13.18 Embedding maps

1.13.19 Retraction and section maps

1.13.20 Continuity

1.13.21 The topology generated by some (open) subsets

1.13.22 Topology bases and sub-bases

1.13.23 Continuity via bases/subbases, hence upper and lower semicontinuity

1.13.24 Pullback topology

definition $pullback_topology::('a\ set) \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('b\ topology) \Rightarrow ('a\ topology)$
where $pullback_topology\ A\ f\ T = topology\ (\lambda S. \exists U. openin\ T\ U \wedge S = f^{-1}U \cap A)$

proposition *continuous_map_pullback* [intro]:

assumes $continuous_map\ T1\ T2\ g$
shows $continuous_map\ (pullback_topology\ A\ f\ T1)\ T2\ (g\ o\ f)$

proposition *continuous_map_pullback'* [intro]:

assumes $continuous_map\ T1\ T2\ (f\ o\ g)$ $topspace\ T1 \subseteq g^{-1}A$
shows $continuous_map\ T1\ (pullback_topology\ A\ f\ T2)\ g$

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1.13.25 Proper maps (not a priori assumed continuous)

1.13.26 Perfect maps (proper, continuous and surjective)

end

1.14 F -Sigma and G -Delta sets in a Topological Space

```
theory FSigma
  imports Abstract_Topology
begin
```

end

1.15 Disjoint sum of arbitrarily many spaces

```
theory Sum_Topology
  imports Abstract_Topology
begin
```

end

Chapter 2

Topology

```
theory Elementary_Topology
imports
  HOL-Library.Set_Idioms
  HOL-Library.Disjoint_Sets
  Product_Vector
begin
```

2.1 Elementary Topology

2.1.1 Topological Basis

```
definition topological_basis  $B \longleftrightarrow$ 
   $(\forall b \in B. \text{open } b) \wedge (\forall x. \text{open } x \longrightarrow (\exists B'. B' \subseteq B \wedge \bigcup B' = x))$ 
```

2.1.2 Countable Basis

```
locale countable_basis = topological_space  $p$  for  $p::'a \text{ set} \Rightarrow \text{bool}$  +
  fixes  $B::'a \text{ set set}$ 
  assumes is_basis: topological_basis  $B$ 
  and countable_basis: countable  $B$ 
begin
```

```
class second_countable_topology = topological_space +
  assumes ex_countable_subbasis:
     $\exists B::'a \text{ set set. countable } B \wedge \text{open} = \text{generate\_topology } B$ 
begin
```

```
proposition Lindelof:
  fixes  $\mathcal{F}::'a::\text{second\_countable\_topology} \text{ set set}$ 
  assumes  $\mathcal{F}: \bigwedge S. S \in \mathcal{F} \Longrightarrow \text{open } S$ 
  obtains  $\mathcal{F}'$  where  $\mathcal{F}' \subseteq \mathcal{F}$  countable  $\mathcal{F}' \cup \mathcal{F}' = \bigcup \mathcal{F}$ 
```

2.1.3 Polish spaces

class *polish_space* = *complete_space* + *second_countable_topology*

2.1.4 Limit Points

definition (in *topological_space*) *islimpt*:: 'a \Rightarrow 'a set \Rightarrow bool (**infixr** *islimpt* 60)
where $x \text{ islimpt } S \iff (\forall T. x \in T \longrightarrow \text{open } T \longrightarrow (\exists y \in S. y \in T \wedge y \neq x))$

2.1.5 Interior of a Set

definition *interior* :: ('a::topological_space) set \Rightarrow 'a set **where**
interior $S = \bigcup \{T. \text{open } T \wedge T \subseteq S\}$

2.1.6 Closure of a Set

definition *closure* :: ('a::topological_space) set \Rightarrow 'a set **where**
closure $S = S \cup \{x . x \text{ islimpt } S\}$

2.1.7 Frontier (also known as boundary)

definition *frontier* :: ('a::topological_space) set \Rightarrow 'a set **where**
frontier $S = \text{closure } S - \text{interior } S$

2.1.8 Limits

2.1.9 Compactness

proposition *Heine_Borel_imp_Bolzano_Weierstrass*:

assumes *compact* S
and *infinite* T
and $T \subseteq S$
shows $\exists x \in S. x \text{ islimpt } T$

definition *countably_compact* :: ('a::topological_space) set \Rightarrow bool **where**
countably_compact $U \iff$

$(\forall A. \text{countable } A \longrightarrow (\forall a \in A. \text{open } a) \longrightarrow U \subseteq \bigcup A$
 $\longrightarrow (\exists T \subseteq A. \text{finite } T \wedge U \subseteq \bigcup T))$

proposition *countably_compact_imp_compact_second_countable*:

countably_compact $U \implies \text{compact } (U :: 'a :: \text{second_countable_topology set})$

definition *seq_compact* :: 'a::topological_space set \Rightarrow bool **where**

$seq_compact\ S \longleftrightarrow$
 $(\forall f. (\forall n. f\ n \in S) \longrightarrow (\exists l \in S. \exists r :: nat \Rightarrow nat. strict_mono\ r \wedge (f \circ r) \longrightarrow l))$

proposition *Bolzano_Weierstrass_imp_seq_compact:*

fixes $S :: 'a :: \{t1_space, first_countable_topology\} set$

shows $(\bigwedge T. \llbracket infinite\ T; T \subseteq S \rrbracket \Longrightarrow \exists x \in S. x\ islimpt\ T) \Longrightarrow seq_compact\ S$

2.1.10 Continuity

2.1.11 Homeomorphisms

definition *homeomorphism* $S\ T\ f\ g \longleftrightarrow$

$(\forall x \in S. (g(f\ x) = x)) \wedge (f\ ' S = T) \wedge continuous_on\ S\ f \wedge$

$(\forall y \in T. (f(g\ y) = y)) \wedge (g\ ' T = S) \wedge continuous_on\ T\ g$

definition *homeomorphic* $:: 'a :: topological_space\ set \Rightarrow 'b :: topological_space\ set \Rightarrow bool$

(infixr *homeomorphic* 60)

where $s\ homeomorphic\ t \equiv (\exists f\ g. homeomorphism\ s\ t\ f\ g)$

end

theory *Abstract_Limits*

imports

Abstract_Topology

begin

2.1.12 nhdsin and atin

2.1.13 Limits in a topological space

2.1.14 Pointwise continuity in topological spaces

2.1.15 Combining theorems for continuous functions into the reals

end

2.2 Non-Denumerability of the Continuum

theory *Continuum_Not_Denumerable*

imports

Complex_Main

HOL-Library.Countable_Set

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begin

theorem *real_non_denum*: $\nexists f :: \text{nat} \Rightarrow \text{real. surj } f$

corollary *complex_non_denum*: $\nexists f :: \text{nat} \Rightarrow \text{complex. surj } f$

end

2.3 Abstract Topology 2

theory *Abstract_Topology_2*

imports

Elementary_Topology Abstract_Topology Continuum_Not_Denumerable

HOL-Library.Indicator_Function

HOL-Library.Equipollence

begin

2.3.1 Closure

corollary *infinite_openin*:

fixes $S :: 'a :: \text{t1_space set}$

shows $\llbracket \text{openin } (\text{top_of_set } U) S; x \in S; x \text{ islimpt } U \rrbracket \Longrightarrow \text{infinite } S$

2.3.2 Frontier

2.3.3 Compactness

2.3.4 Continuity

2.3.5 Retractions

definition *retraction* :: $('a :: \text{topological_space}) \text{ set} \Rightarrow 'a \text{ set} \Rightarrow ('a \Rightarrow 'a) \Rightarrow \text{bool}$

where *retraction* $S T r \longleftrightarrow$

$T \subseteq S \wedge \text{continuous_on } S r \wedge r \in S \rightarrow T \wedge (\forall x \in T. r x = x)$

definition *retract_of* (**infixl** *retract'_of* 50) **where**

$T \text{ retract_of } S \longleftrightarrow (\exists r. \text{retraction } S T r)$

2.3.6 Retractions on a topological space

2.3.7 Paths and path-connectedness

2.3.8 Connected components

2.3.9 Combining theorems for continuous functions into the reals

2.3.10 A few cardinality results

end

2.4 Connected Components

```
theory Connected
  imports
    Abstract_Topology_2
begin
```

2.4.1 Connected components, considered as a connectedness relation or a set

definition *connected_component* $S\ x\ y \equiv \exists T. \text{connected } T \wedge T \subseteq S \wedge x \in T \wedge y \in T$

2.4.2 The set of connected components of a set

definition *components*:: $'a::\text{topological_space set} \Rightarrow 'a\ \text{set set}$
where *components* $S \equiv \text{connected_component_set } S\ 'S$

2.4.3 Lemmas about components

proposition *component_diff_connected*:
fixes $S :: 'a::\text{metric_space set}$
assumes *connected* S *connected* $U\ S \subseteq U$ **and** $C: C \in \text{components } (U - S)$
shows *connected* $(U - C)$

end

```
theory Function_Topology
  imports
    Elementary_Topology
```

Abstract Limits
Connected

begin

2.5 Function Topology

2.5.1 The product topology

definition *product_topology*::('i \Rightarrow ('a topology)) \Rightarrow ('i set) \Rightarrow (('i \Rightarrow 'a) topology)
where *product_topology* T I =
topology_generated_by {($\Pi_E i \in I. X i$) | X. ($\forall i. \text{openin } (T i) (X i)$) \wedge finite {i. X i \neq topspace (T i)}}}

proposition *product_topology*:

product_topology X I =
topology
 (arbitrary union_of
 ((finite intersection_of
 ($\lambda F. \exists i U. F = \{f. f i \in U\} \wedge i \in I \wedge \text{openin } (X i) U$)
 relative_to ($\Pi_E i \in I. \text{topspace } (X i)$))))
 (is _ = topology (_ union_of ((_ intersection_of ? Ψ) relative_to ?TOP)))

proposition *product_topology_open_contains_basis*:

assumes *openin* (product_topology T I) U x \in U
shows $\exists X. x \in (\Pi_E i \in I. X i) \wedge (\forall i. \text{openin } (T i) (X i)) \wedge \text{finite } \{i. X i \neq \text{topspace } (T i)\} \wedge (\Pi_E i \in I. X i) \subseteq U$

corollary *openin_product_topology_alt*:

openin (product_topology X I) S \longleftrightarrow
 ($\forall x \in S. \exists U. \text{finite } \{i \in I. U i \neq \text{topspace}(X i)\} \wedge$
 ($\forall i \in I. \text{openin } (X i) (U i)$) $\wedge x \in \text{PiE } I U \wedge \text{PiE } I U \subseteq S$)

corollary *closedin_product_topology*:

closedin (product_topology X I) (PiE I S) \longleftrightarrow PiE I S = {} \vee ($\forall i \in I. \text{closedin } (X i) (S i)$)

corollary *closedin_product_topology_singleton*:

f \in extensional I \implies *closedin* (product_topology X I) {f} \longleftrightarrow ($\forall i \in I. \text{closedin } (X i) \{f i\}$)

Powers of a single topological space as a topological space, using type classes

instantiation *fun* :: (type, topological_space) topological_space
begin

definition *open_fun_def*:

open U = *openin* (product_topology ($\lambda i. \text{euclidean}$) UNIV) U

proposition *product_topology_basis'*:
fixes $x::'i \Rightarrow 'a$ **and** $U::'i \Rightarrow ('b::\text{topological_space})$ *set*
assumes $\text{finite } I \wedge i. i \in I \implies \text{open } (U\ i)$
shows $\text{open } \{f. \forall i \in I. f\ (x\ i) \in U\ i\}$

Topological countability for product spaces

proposition *product_topology_countable_basis*:
shows $\exists K::('a::\text{countable} \Rightarrow 'b::\text{second_countable_topology})$ *set set*).
 $\text{topological_basis } K \wedge \text{countable } K \wedge$
 $(\forall k \in K. \exists X. (k = \text{PiE } \text{UNIV } X) \wedge (\forall i. \text{open } (X\ i)) \wedge \text{finite } \{i. X\ i \neq \text{UNIV}\})$

2.5.2 The Alexander subbase theorem

theorem *Alexander_subbase*:
assumes $X: \text{topology } (\text{arbitrary_union_of } (\text{finite_intersection_of } (\lambda x. x \in \mathcal{B})) \text{ relative_to } \bigcup \mathcal{B})) = X$
and $\text{fin}: \bigwedge C. \llbracket C \subseteq \mathcal{B}; \bigcup C = \text{topspace } X \rrbracket \implies \exists C'. \text{finite } C' \wedge C' \subseteq C \wedge \bigcup C' = \text{topspace } X$
shows *compact_space* X

corollary *Alexander_subbase_alt*:
assumes $U \subseteq \bigcup \mathcal{B}$
and $\text{fin}: \bigwedge C. \llbracket C \subseteq \mathcal{B}; U \subseteq \bigcup C \rrbracket \implies \exists C'. \text{finite } C' \wedge C' \subseteq C \wedge U \subseteq \bigcup C'$
and $X: \text{topology } (\text{arbitrary_union_of } (\text{finite_intersection_of } (\lambda x. x \in \mathcal{B}) \text{ relative_to } U)) = X$
shows *compact_space* X

proposition *continuous_map_componentwise*:
 $\text{continuous_map } X (\text{product_topology } Y\ I) f \longleftrightarrow$
 $f\ ' (\text{topspace } X) \subseteq \text{extensional } I \wedge (\forall k \in I. \text{continuous_map } X (Y\ k) (\lambda x. f\ x\ k))$
(is ?lhs \longleftrightarrow $_ \wedge$ **?rhs)**

proposition *open_map_product_projection*:
assumes $i \in I$
shows $\text{open_map } (\text{product_topology } Y\ I) (Y\ i) (\lambda f. f\ i)$

2.5.3 Open Pi-sets in the product topology

proposition *openin_PiE_gen*:
 $\text{openin } (\text{product_topology } X\ I) (\text{PiE } I\ S) \longleftrightarrow$

$$PiE\ I\ S = \{\} \vee$$

$$finite\ \{i \in I.\ S\ i \neq\ topspace\ (X\ i)\} \wedge (\forall i \in I.\ openin\ (X\ i)\ (S\ i))$$
(is ?lhs \longleftrightarrow _ \vee ?rhs)

corollary *openin_PiE:*

$$finite\ I \implies openin\ (product_topology\ X\ I)\ (PiE\ I\ S) \longleftrightarrow PiE\ I\ S = \{\} \vee (\forall i \in I.\ openin\ (X\ i)\ (S\ i))$$

proposition *compact_space_product_topology:*

$$compact_space(product_topology\ X\ I) \longleftrightarrow$$

$$(product_topology\ X\ I) = trivial_topology \vee (\forall i \in I.\ compact_space(X\ i))$$
(is ?lhs = ?rhs)

corollary *compactin_PiE:*

$$compactin\ (product_topology\ X\ I)\ (PiE\ I\ S) \longleftrightarrow$$

$$PiE\ I\ S = \{\} \vee (\forall i \in I.\ compactin\ (X\ i)\ (S\ i))$$

2.5.4 Relationship with connected spaces, paths, etc.

proposition *connected_space_product_topology:*

$$connected_space(product_topology\ X\ I) \longleftrightarrow$$

$$(\exists i \in I.\ X\ i = trivial_topology) \vee (\forall i \in I.\ connected_space(X\ i))$$
(is ?lhs \longleftrightarrow ?eq \vee ?rhs)

2.5.5 Projections from a function topology to a component

2.5.6 Limits

end

2.6 The binary product topology

theory *Product_Topology*
imports *Function_Topology*
begin

2.7 Product Topology

2.7.1 Definition

2.7.2 Continuity

proposition *compact_space_prod_topology:*

$compact_space(prod_topology\ X\ Y) \longleftrightarrow (prod_topology\ X\ Y) = trivial_topology$
 $\vee compact_space\ X \wedge compact_space\ Y$

2.7.3 Homeomorphic maps

proposition *connected_space_prod_topology:*

$connected_space(prod_topology\ X\ Y) \longleftrightarrow$
 $(prod_topology\ X\ Y) = trivial_topology \vee connected_space\ X \wedge connected_space$
 Y (is ?lhs=?rhs)

end

2.8 T1 and Hausdorff spaces

theory *T1_Spaces*
imports *Product_Topology*
begin

2.9 T1 spaces with equivalences to many naturally "nice" properties.

proposition *t1_space_product_topology:*

$t1_space\ (product_topology\ X\ I)$
 $\longleftrightarrow (product_topology\ X\ I) = trivial_topology \vee (\forall i \in I. t1_space\ (X\ i))$

2.9.1 Hausdorff Spaces

end

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2.10 Lindelöf spaces

```
theory Lindelof_Spaces  
imports T1_Spaces  
begin
```

```
end
```

Chapter 3

Functional Analysis

```
theory Metric_Arith
  imports HOL.Real_Vector_Spaces
begin
theorem metric_eq_thm [THEN HOL.eq_reflection]:
   $x \in s \implies y \in s \implies x = y \longleftrightarrow (\forall a \in s. \text{dist } x \ a = \text{dist } y \ a)$ 
end
```


Chapter 4

Elementary Metric Spaces

```
theory Elementary_Metric_Spaces
imports
  Abstract_Topology_2
  Metric_Arith
begin
```

4.1 Open and closed balls

```
definition ball :: 'a::metric_space  $\Rightarrow$  real  $\Rightarrow$  'a set
where ball x e = {y. dist x y < e}
```

```
definition cball :: 'a::metric_space  $\Rightarrow$  real  $\Rightarrow$  'a set
where cball x e = {y. dist x y  $\leq$  e}
```

```
definition sphere :: 'a::metric_space  $\Rightarrow$  real  $\Rightarrow$  'a set
where sphere x e = {y. dist x y = e}
```

4.2 Limit Points

4.3 Perfect Metric Spaces

4.4 Finite and discrete

4.5 Interior

4.6 Frontier

4.7 Limits

```
proposition Lim: (f  $\longrightarrow$  l) net  $\iff$  trivial_limit_net  $\vee$  ( $\forall$  e>0. eventually ( $\lambda$ x.  
dist (f x) l < e) net)
```

proposition *Lim_within_le*: $(f \longrightarrow l)(\text{at } a \text{ within } S) \longleftrightarrow$
 $(\forall e > 0. \exists d > 0. \forall x \in S. 0 < \text{dist } x \ a \wedge \text{dist } x \ a \leq d \longrightarrow \text{dist } (f \ x) \ l < e)$

proposition *Lim_within*: $(f \longrightarrow l) (\text{at } a \text{ within } S) \longleftrightarrow$
 $(\forall e > 0. \exists d > 0. \forall x \in S. 0 < \text{dist } x \ a \wedge \text{dist } x \ a < d \longrightarrow \text{dist } (f \ x) \ l < e)$

corollary *Lim_withinI* [*intro?*]:

assumes $\bigwedge e. e > 0 \implies \exists d > 0. \forall x \in S. 0 < \text{dist } x \ a \wedge \text{dist } x \ a < d \longrightarrow \text{dist}$
 $(f \ x) \ l \leq e$
shows $(f \longrightarrow l) (\text{at } a \text{ within } S)$

proposition *Lim_at*: $(f \longrightarrow l) (\text{at } a) \longleftrightarrow$
 $(\forall e > 0. \exists d > 0. \forall x. 0 < \text{dist } x \ a \wedge \text{dist } x \ a < d \longrightarrow \text{dist } (f \ x) \ l < e)$

4.8 Continuity

proposition *continuous_within_eps_delta*:

continuous $(\text{at } x \text{ within } s) \ f \longleftrightarrow (\forall e > 0. \exists d > 0. \forall x' \in s. \text{dist } x' \ x < d \longrightarrow$
 $\text{dist } (f \ x') \ (f \ x) < e)$

corollary *continuous_at_eps_delta*:

continuous $(\text{at } x) \ f \longleftrightarrow (\forall e > 0. \exists d > 0. \forall x'. \text{dist } x' \ x < d \longrightarrow \text{dist } (f \ x') \ (f$
 $x) < e)$

4.9 Closure and Limit Characterization

4.10 Boundedness

definition (*in metric_space*) *bounded* $:: 'a \text{ set} \implies \text{bool}$

where *bounded* $S \longleftrightarrow (\exists x \ e. \forall y \in S. \text{dist } x \ y \leq e)$

4.11 Compactness

proposition *seq_compact_imp_totally_bounded*:

assumes *seq_compact* S

shows $\forall e > 0. \exists k. \text{finite } k \wedge k \subseteq S \wedge S \subseteq (\bigcup x \in k. \text{ball } x \ e)$

proposition *seq_compact_imp_Heine_Borel*:

fixes $S :: 'a :: \text{metric_space set}$

assumes *seq_compact* S

shows *compact* S

proposition *compact_eq_seq_compact_metric*:

compact $(S :: 'a :: \text{metric_space set}) \longleftrightarrow \text{seq_compact } S$

proposition *compact_def*: — this is the definition of compactness in HOL Light
 $compact (S :: 'a::metric_space\ set) \longleftrightarrow$
 $(\forall f. (\forall n. f\ n \in S) \longrightarrow (\exists l \in S. \exists r::nat \Rightarrow nat. strict_mono\ r \wedge (f \circ r) \longrightarrow l))$

proposition *compact_eq_Bolzano_Weierstrass*:
fixes $S :: 'a::metric_space\ set$
shows $compact\ S \longleftrightarrow (\forall T. infinite\ T \wedge T \subseteq S \longrightarrow (\exists x \in S. x\ islimpt\ T))$

proposition *Bolzano_Weierstrass_imp_bounded*:
 $(\bigwedge T. [infinite\ T; T \subseteq S]) \Longrightarrow (\exists x \in S. x\ islimpt\ T) \Longrightarrow bounded\ S$

4.12 Banach fixed point theorem

theorem *banach_fix*:— TODO: rename to *Banach_fix*
assumes $s: complete\ s\ s \neq \{\}$
and $c: 0 \leq c < 1$
and $f: f\ 's \subseteq s$
and *lipschitz*: $\forall x \in s. \forall y \in s. dist\ (f\ x)\ (f\ y) \leq c * dist\ x\ y$
shows $\exists! x \in s. f\ x = x$

4.13 Edelstein fixed point theorem

theorem *Edelstein_fix*:
fixes $S :: 'a::metric_space\ set$
assumes $S: compact\ S\ S \neq \{\}$
and $gs: (g\ 'S) \subseteq S$
and *dist*: $\forall x \in S. \forall y \in S. x \neq y \longrightarrow dist\ (g\ x)\ (g\ y) < dist\ x\ y$
shows $\exists! x \in S. g\ x = x$

4.14 The diameter of a set

definition *diameter* :: $'a::metric_space\ set \Rightarrow real$ **where**
 $diameter\ S = (if\ S = \{\} then\ 0\ else\ SUP\ (x,y) \in S \times S. dist\ x\ y)$

proposition *Lebesgue_number_lemma*:
assumes $compact\ S\ C \neq \{\}$ $S \subseteq \bigcup C$ **and** *ope*: $\bigwedge B. B \in C \Longrightarrow open\ B$
obtains δ **where** $0 < \delta \wedge T. [T \subseteq S; diameter\ T < \delta] \Longrightarrow \exists B \in C. T \subseteq B$

4.15 Metric spaces with the Heine-Borel property

class *heine_borel* = *metric_space* +
assumes *bounded_imp_convergent_subsequence*:

bounded (*range f*) $\implies \exists l r. \text{strict_mono } (r::\text{nat}\Rightarrow\text{nat}) \wedge ((f \circ r) \longrightarrow l)$
sequentially

proposition *bounded_closed_imp_seq_compact*:

fixes *S* :: 'a::heine_borel set

assumes *bounded S*

and *closed S*

shows *seq_compact S*

instance *real* :: heine_borel

instance *prod* :: (heine_borel, heine_borel) heine_borel

4.16 Completeness

proposition (in *metric_space*) *completeI*:

assumes $\bigwedge f. \forall n. f\ n \in s \implies \text{Cauchy } f \implies \exists l \in s. f \longrightarrow l$

shows *complete s*

proposition (in *metric_space*) *completeE*:

assumes *complete s* **and** $\forall n. f\ n \in s$ **and** *Cauchy f*

obtains *l* **where** $l \in s$ **and** $f \longrightarrow l$

proposition *compact_eq_totally_bounded*:

compact s \longleftrightarrow *complete s* $\wedge (\forall e > 0. \exists k. \text{finite } k \wedge s \subseteq (\bigcup x \in k. \text{ball } x\ e))$

(**is** $_ \longleftrightarrow$ *?rhs*)

4.17 Cauchy continuity

4.18 Properties of Balls and Spheres

4.19 Distance from a Set

4.20 Infimum Distance

definition *infdist x A* = (if $A = \{\}$ then 0 else $\text{INF } a \in A. \text{dist } x\ a$)

4.21 Separation between Points and Sets

proposition *separate_point_closed*:

fixes *S* :: 'a::heine_borel set

assumes *closed S* **and** $a \notin S$

shows $\exists d > 0. \forall x \in S. d \leq \text{dist } a\ x$

proposition *separate_compact_closed*:
fixes $S T :: 'a::\text{heine_borel_set}$
assumes $\text{compact } S$
and $T: \text{closed } T \ S \cap T = \{\}$
shows $\exists d > 0. \forall x \in S. \forall y \in T. d \leq \text{dist } x \ y$

proposition *separate_closed_compact*:
fixes $S T :: 'a::\text{heine_borel_set}$
assumes $S: \text{closed } S$
and $T: \text{compact } T$
and $\text{dis}: S \cap T = \{\}$
shows $\exists d > 0. \forall x \in S. \forall y \in T. d \leq \text{dist } x \ y$

proposition *compact_in_open_separated*:
fixes $A :: 'a::\text{heine_borel_set}$
assumes $A: A \neq \{\}$ $\text{compact } A$
assumes $\text{open } B$
assumes $A \subseteq B$
obtains e **where** $e > 0 \ \{x. \text{infdist } x \ A \leq e\} \subseteq B$

4.22 Uniform Continuity

4.23 Continuity on a Compact Domain Implies Uniform Continuity

corollary *compact_uniformly_continuous*:
fixes $f :: 'a :: \text{metric_space} \Rightarrow 'b :: \text{metric_space}$
assumes $f: \text{continuous_on } S \ f$ **and** $S: \text{compact } S$
shows $\text{uniformly_continuous_on } S \ f$

4.24 With Abstract Topology (TODO: move and remove dependency?)

4.25 Closed Nest

4.26 Consequences for Real Numbers

4.27 The infimum of the distance between two sets

definition *setdist* $:: 'a::\text{metric_space} \ \text{set} \Rightarrow 'a \ \text{set} \Rightarrow \text{real}$ **where**
 $\text{setdist } s \ t \equiv$
 $(\text{if } s = \{\} \vee t = \{\} \text{ then } 0$

else Inf {dist x y | x y. x ∈ s ∧ y ∈ t})

proposition *setdist_attains_inf*:

assumes *compact B B ≠ {}*

obtains y where *y ∈ B setdist A B = infdist y A*

4.28 Diameter Lemma

end

4.29 Elementary Normed Vector Spaces

theory *Elementary_Normed_Spaces*

imports

HOL-Library.FuncSet

Elementary_Metric_Spaces Cartesian_Space

Connected

begin

4.29.1 Orthogonal Transformation of Balls

4.29.2 Support

4.29.3 Intervals

4.29.4 Limit Points

4.29.5 Balls and Spheres in Normed Spaces

corollary *compact_sphere [simp]*:

fixes *a :: 'a::{real_normed_vector,perfect_space,heine_borel}*

shows *compact (sphere a r)*

corollary *bounded_sphere [simp]*:

fixes *a :: 'a::{real_normed_vector,perfect_space,heine_borel}*

shows *bounded (sphere a r)*

corollary *closed_sphere [simp]*:

fixes *a :: 'a::{real_normed_vector,perfect_space,heine_borel}*

shows *closed (sphere a r)*

4.29.6 Filters

4.29.7 Trivial Limits

4.29.8 Limits

proposition *Lim_at_infinity*: $(f \longrightarrow l) \text{ at_infinity} \longleftrightarrow (\forall e > 0. \exists b. \forall x. \text{norm } x \geq b \longrightarrow \text{dist } (f x) l < e)$

corollary *Lim_at_infinityI* [*intro?*]:

assumes $\bigwedge e. e > 0 \implies \exists B. \forall x. \text{norm } x \geq B \longrightarrow \text{dist } (f x) l \leq e$
shows $(f \longrightarrow l) \text{ at_infinity}$

4.29.9 Boundedness

corollary *cobounded_imp_unbounded*:

fixes $S :: 'a::\{\text{real_normed_vector}, \text{perfect_space}\} \text{ set}$
shows $\text{bounded } (- S) \implies \neg \text{bounded } S$

4.29.10 Normed spaces with the Heine-Borel property

4.29.11 Intersecting chains of compact sets and the Baire property

proposition *bounded_closed_chain*:

fixes $\mathcal{F} :: 'a::\text{heine_borel} \text{ set set}$
assumes $B \in \mathcal{F} \text{ bounded } B \text{ and } \mathcal{F}: \bigwedge S. S \in \mathcal{F} \implies \text{closed } S \text{ and } \{\} \notin \mathcal{F}$
and chain: $\bigwedge S T. S \in \mathcal{F} \wedge T \in \mathcal{F} \implies S \subseteq T \vee T \subseteq S$
shows $\bigcap \mathcal{F} \neq \{\}$

corollary *compact_chain*:

fixes $\mathcal{F} :: 'a::\text{heine_borel} \text{ set set}$
assumes $\bigwedge S. S \in \mathcal{F} \implies \text{compact } S \{\} \notin \mathcal{F}$
 $\bigwedge S T. S \in \mathcal{F} \wedge T \in \mathcal{F} \implies S \subseteq T \vee T \subseteq S$
shows $\bigcap \mathcal{F} \neq \{\}$

theorem *Baire*:

fixes $S::'a::\{\text{real_normed_vector}, \text{heine_borel}\} \text{ set}$
assumes $\text{closed } S \text{ countable } \mathcal{G}$
and ope: $\bigwedge T. T \in \mathcal{G} \implies \text{openin } (\text{top_of_set } S) T \wedge S \subseteq \text{closure } T$
shows $S \subseteq \text{closure}(\bigcap \mathcal{G})$

4.29.12 Continuity

proposition *homeomorphic_ball_UNIV*:

fixes $a :: 'a::\text{real_normed_vector}$

assumes $0 < r$ shows ball a r homeomorphic (UNIV:: 'a set)

4.29.13 Connected Normed Spaces

end

4.30 Linear Decision Procedure for Normed Spaces

theory *Norm_Arith*

imports *HOL-Library.Sum_of_Squares*

begin

method_setup *norm* = <

Scan.succeed (SIMPLE_METHOD' o NormArith.norm_arith_tac)

> *prove simple linear statements about vector norms*

proposition *dist_triangle_add*:

fixes $x\ y\ x'\ y' :: 'a::\text{real_normed_vector}$

shows $\text{dist } (x + y) (x' + y') \leq \text{dist } x\ x' + \text{dist } y\ y'$

end

Chapter 5

Vector Analysis

```
theory Topology_Euclidean_Space
  imports
    Elementary_Normed_Spaces
    Linear_Algebra
    Norm_Arith
begin
```

5.1 Elementary Topology in Euclidean Space

5.1.1 Boxes

```
abbreviation One :: 'a::euclidean_space where
  One  $\equiv \sum Basis$ 
```

```
definition (in euclidean_space) eucl_less (infix <e 50) where
  eucl_less a b  $\longleftrightarrow (\forall i \in Basis. a \cdot i < b \cdot i)$ 
```

```
definition box_eucl_less: box a b = {x. a <e x  $\wedge$  x <e b}
```

```
definition cbox a b = {x.  $\forall i \in Basis. a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i$ }
```

```
corollary open_countable_Union_open_box:
```

```
  fixes S :: 'a :: euclidean_space set
  assumes open S
  obtains D where countable D  $D \subseteq Pow S \wedge X. X \in D \implies \exists a b. X = box a b$ 
 $\bigcup D = S$ 
```

```
corollary open_countable_Union_open_cbox:
```

```
  fixes S :: 'a :: euclidean_space set
  assumes open S
  obtains D where countable D  $D \subseteq Pow S \wedge X. X \in D \implies \exists a b. X = cbox a$ 
 $b \bigcup D = S$ 
```

5.1.2 General Intervals

definition *is_interval* ($s :: ('a :: euclidean_space) set$) \longleftrightarrow
 $(\forall a \in s. \forall b \in s. \forall x. (\forall i \in \text{Basis}. ((a \cdot i \leq x \cdot i \wedge x \cdot i \leq b \cdot i) \vee (b \cdot i \leq x \cdot i \wedge x \cdot i \leq a \cdot i)))$
 $\longrightarrow x \in s)$

5.1.3 Limit Component Bounds

5.1.4 Class Instances

instance *euclidean_space* \subseteq *heine_borel*

instance *euclidean_space* \subseteq *banach*

5.1.5 Compact Boxes

proposition *is_interval_compact*:
 $is_interval\ S \wedge compact\ S \longleftrightarrow (\exists a\ b. S = cbox\ a\ b) \quad (\text{is ?lhs} = ?rhs)$

proposition *tendsto_componentwise_iff*:
fixes $f :: _ \Rightarrow 'b :: euclidean_space$
shows $(f \longrightarrow l)\ F \longleftrightarrow (\forall i \in \text{Basis}. ((\lambda x. (f\ x \cdot i)) \longrightarrow (l \cdot i))\ F)$
 $(\text{is ?lhs} = ?rhs)$

corollary *continuous_componentwise*:
 $continuous\ F\ f \longleftrightarrow (\forall i \in \text{Basis}. continuous\ F\ (\lambda x. (f\ x \cdot i)))$

corollary *continuous_on_componentwise*:
fixes $S :: 'a :: t2_space\ set$
shows $continuous_on\ S\ f \longleftrightarrow (\forall i \in \text{Basis}. continuous_on\ S\ (\lambda x. (f\ x \cdot i)))$

5.1.6 Separability

proposition *separable*:
fixes $S :: 'a :: \{metric_space, second_countable_topology\}\ set$
obtains T **where** $countable\ T\ T \subseteq S\ S \subseteq closure\ T$

proposition *open_surjective_linear_image*:
fixes $f :: 'a :: real_normed_vector \Rightarrow 'b :: euclidean_space$
assumes $open\ A\ linear\ f\ surj\ f$
shows $open(f\ 'A)$

corollary *open_bijjective_linear_image_eq*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *linear f bij f*
shows $open(f \text{ ` } A) \longleftrightarrow open A$

corollary *interior_bijjective_linear_image*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes *linear f bij f*
shows $interior (f \text{ ` } S) = f \text{ ` } interior S$

proposition *injective_imp_isometric*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $s: closed\ s\ subspace\ s$
and $f: bounded_linear\ f\ \forall x \in s. f\ x = 0 \longrightarrow x = 0$
shows $\exists e > 0. \forall x \in s. norm (f\ x) \geq e * norm\ x$

proposition *closed_injective_image_subspace*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $subspace\ s\ bounded_linear\ f\ \forall x \in s. f\ x = 0 \longrightarrow x = 0\ closed\ s$
shows $closed(f \text{ ` } s)$

5.1.7 Set Distance

corollary *setdist_gt_0_compact_closed*:
assumes $S: compact\ S$ **and** $T: closed\ T$
shows $setdist\ S\ T > 0 \longleftrightarrow (S \neq \{\} \wedge T \neq \{\} \wedge S \cap T = \{\})$

end

5.2 Line Segment

theory *Line_Segment*
imports
 Convex
 Topology_Euclidean_Space
begin

corollary *component_complement_connected*:
fixes $S :: 'a::real_normed_vector\ set$
assumes $connected\ S\ C \in components\ (-S)$
shows $connected(-C)$

proposition *clopen*:
fixes $S :: 'a :: real_normed_vector\ set$
shows $closed\ S \wedge open\ S \longleftrightarrow S = \{\} \vee S = UNIV$

corollary *compact_open*:

fixes $S :: 'a :: euclidean_space$ set
shows $compact\ S \wedge open\ S \longleftrightarrow S = \{\}$

corollary *finite_imp_not_open*:
fixes $S :: 'a :: \{real_normed_vector, perfect_space\}$ set
shows $\llbracket finite\ S; open\ S \rrbracket \Longrightarrow S = \{\}$

corollary *empty_interior_finite*:
fixes $S :: 'a :: \{real_normed_vector, perfect_space\}$ set
shows $finite\ S \Longrightarrow interior\ S = \{\}$

5.2.1 Midpoint

definition *midpoint* $:: 'a :: real_vector \Rightarrow 'a \Rightarrow 'a$
where $midpoint\ a\ b = (inverse\ (2::real)) *_{\mathbb{R}} (a + b)$

5.2.2 Open and closed segments

definition *closed_segment* $:: 'a :: real_vector \Rightarrow 'a \Rightarrow 'a$ set
where $closed_segment\ a\ b = \{(1 - u) *_{\mathbb{R}} a + u *_{\mathbb{R}} b \mid u::real. 0 \leq u \wedge u \leq 1\}$

definition *open_segment* $:: 'a :: real_vector \Rightarrow 'a \Rightarrow 'a$ set **where**
 $open_segment\ a\ b \equiv closed_segment\ a\ b - \{a, b\}$

proposition *dist_decreases_open_segment*:
fixes $a :: 'a :: euclidean_space$
assumes $x \in open_segment\ a\ b$
shows $dist\ c\ x < dist\ c\ a \vee dist\ c\ x < dist\ c\ b$

corollary *open_segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a :: euclidean_space$
assumes $x \in open_segment\ a\ b$
shows $norm\ (y - x) < norm\ (y - a) \vee norm\ (y - x) < norm\ (y - b)$

corollary *dist_decreases_closed_segment*:
fixes $a :: 'a :: euclidean_space$
assumes $x \in closed_segment\ a\ b$
shows $dist\ c\ x \leq dist\ c\ a \vee dist\ c\ x \leq dist\ c\ b$

corollary *segment_furthest_le*:
fixes $a\ b\ x\ y :: 'a :: euclidean_space$
assumes $x \in closed_segment\ a\ b$
shows $norm\ (y - x) \leq norm\ (y - a) \vee norm\ (y - x) \leq norm\ (y - b)$

5.2.3 Betweenness

definition $between = (\lambda(a,b) x. x \in closed_segment\ a\ b)$

end

5.3 Convex Sets and Functions on (Normed) Euclidean Spaces

theory *Convex_Euclidean_Space*

imports

Convex Topology_Euclidean_Space Line_Segment

begin

corollary *empty_interior_lowdim:*

fixes $S :: 'n::euclidean_space\ set$

shows $dim\ S < DIM\ ('n) \implies interior\ S = \{\}$

corollary *aff_dim_nonempty_interior:*

fixes $S :: 'a::euclidean_space\ set$

shows $interior\ S \neq \{\} \implies aff_dim\ S = DIM('a)$

5.3.1 Relative interior of a set

definition $rel_interior\ S =$

$\{x. \exists T. openin\ (top_of_set\ (affine\ hull\ S))\ T \wedge x \in T \wedge T \subseteq S\}$

definition $rel_open\ S \longleftrightarrow rel_interior\ S = S$

5.3.2 Closest point of a convex set is unique, with a continuous projection

definition $closest_point :: 'a::\{real_inner,heine_borel\}\ set \Rightarrow 'a \Rightarrow 'a$

where $closest_point\ S\ a = (SOME\ x. x \in S \wedge (\forall y \in S. dist\ a\ x \leq dist\ a\ y))$

proposition *closest_point_in_rel_interior:*

assumes $closed\ S\ S \neq \{\}$ **and** $x: x \in affine\ hull\ S$

shows $closest_point\ S\ x \in rel_interior\ S \longleftrightarrow x \in rel_interior\ S$

end

Chapter 6

Unsorted

```
theory Starlike
imports
  Convex_Euclidean_Space
  Line_Segment
begin
```

6.0.1 The relative frontier of a set

definition $rel_frontier\ S = closure\ S - rel_interior\ S$

proposition *ray_to_rel_frontier*:
fixes $a :: 'a::real_inner$
assumes $bounded\ S$
 and $a: a \in rel_interior\ S$
 and $aff: (a + l) \in affine\ hull\ S$
 and $l \neq 0$
obtains d **where** $0 < d$ $(a + d *_R l) \in rel_frontier\ S$
 $\wedge e. \llbracket 0 \leq e; e < d \rrbracket \implies (a + e *_R l) \in rel_interior\ S$

corollary *ray_to_frontier*:
fixes $a :: 'a::euclidean_space$
assumes $bounded\ S$
 and $a: a \in interior\ S$
 and $l \neq 0$
obtains d **where** $0 < d$ $(a + d *_R l) \in frontier\ S$
 $\wedge e. \llbracket 0 \leq e; e < d \rrbracket \implies (a + e *_R l) \in interior\ S$

proposition *rel_frontier_not_sing*:
fixes $a :: 'a::euclidean_space$
assumes $bounded\ S$
 shows $rel_frontier\ S \neq \{a\}$

6.0.2 Coplanarity, and collinearity in terms of affine hull

definition *coplanar* **where**

$$\text{coplanar } S \equiv \exists u \ v \ w. S \subseteq \text{affine hull } \{u, v, w\}$$

6.0.3 Connectedness of the intersection of a chain

proposition *connected_chain*:

fixes $\mathcal{F} :: 'a :: \text{euclidean_space set set}$

assumes $cc: \bigwedge S. S \in \mathcal{F} \implies \text{compact } S \wedge \text{connected } S$

and linear: $\bigwedge S \ T. S \in \mathcal{F} \wedge T \in \mathcal{F} \implies S \subseteq T \vee T \subseteq S$

shows $\text{connected}(\bigcap \mathcal{F})$

6.0.4 Proper maps, including projections out of compact sets

proposition *proper_map*:

fixes $f :: 'a :: \text{heine_borel} \Rightarrow 'b :: \text{heine_borel}$

assumes $\text{closedin } (\text{top_of_set } S) \ K$

and com: $\bigwedge U. [U \subseteq T; \text{compact } U] \implies \text{compact } (S \cap f^{-1} U)$

and $f^{-1} S \subseteq T$

shows $\text{closedin } (\text{top_of_set } T) (f^{-1} K)$

6.0.5 Closure of conic hulls

proposition *closedin_conic_hull*:

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $\text{compact } T \ 0 \notin T \ T \subseteq S$

shows $\text{closedin } (\text{top_of_set } (\text{conic hull } S)) (\text{conic hull } T)$

corollary *affine_hull_convex_Int_open*:

fixes $S :: 'a :: \text{real_normed_vector set}$

assumes $\text{convex } S \ \text{open } T \ S \cap T \neq \{\}$

shows $\text{affine hull } (S \cap T) = \text{affine hull } S$

corollary *affine_hull_affine_Int_nonempty_interior*:

fixes $S :: 'a :: \text{real_normed_vector set}$

assumes $\text{affine } S \ S \cap \text{interior } T \neq \{\}$

shows $\text{affine hull } (S \cap T) = \text{affine hull } S$

corollary *affine_hull_affine_Int_open*:

fixes $S :: 'a :: \text{real_normed_vector set}$

assumes *affine S open T S* $S \cap T \neq \{\}$
shows *affine hull (S* $\cap T) = \text{affine hull } S$

corollary *affine_hull_convex_Int_openin*:

fixes *S* :: 'a::real_normed_vector set
assumes *convex S openin (top_of_set (affine hull S)) T S* $S \cap T \neq \{\}$
shows *affine hull (S* $\cap T) = \text{affine hull } S$

corollary *affine_hull_openin*:

fixes *S* :: 'a::real_normed_vector set
assumes *openin (top_of_set (affine hull T)) S S* $S \neq \{\}$
shows *affine hull S = affine hull T*

corollary *affine_hull_open*:

fixes *S* :: 'a::real_normed_vector set
assumes *open S S* $S \neq \{\}$
shows *affine hull S = UNIV*

proposition *aff_dim_eq_hyperplane*:

fixes *S* :: 'a::euclidean_space set
shows *aff_dim S = DIM('a) - 1* $\longleftrightarrow (\exists a b. a \neq 0 \wedge \text{affine hull } S = \{x. a \cdot x = b\})$
(is ?lhs = ?rhs)

corollary *aff_dim_hyperplane [simp]*:

fixes *a* :: 'a::euclidean_space
shows *a* $\neq 0 \implies \text{aff_dim } \{x. a \cdot x = r\} = \text{DIM}('a) - 1$

proposition *aff_dim_sums_Int*:

assumes *affine S*
and *affine T*
and *S* $\cap T \neq \{\}$
shows *aff_dim {x + y | x y. x* $\in S \wedge y \in T\} = (\text{aff_dim } S + \text{aff_dim } T) - \text{aff_dim}(S \cap T)$

6.0.6 Lower-dimensional affine subsets are nowhere dense

proposition *dense_complement_subspace*:

fixes *S* :: 'a :: euclidean_space set
assumes *dim_less: dim T < dim S* **and** *subspace S* **shows** *closure(S - T) = S*

6.0.7 Paracompactness

proposition *paracompact*:

fixes $S :: 'a :: \{\text{metric_space}, \text{second_countable_topology}\}$ set

assumes $S \subseteq \bigcup C$ and $opC: \bigwedge T. T \in C \implies \text{open } T$

obtains C' where $S \subseteq \bigcup C'$

and $\bigwedge U. U \in C' \implies \text{open } U \wedge (\exists T. T \in C \wedge U \subseteq T)$

and $\bigwedge x. x \in S$

$\implies \exists V. \text{open } V \wedge x \in V \wedge \text{finite } \{U. U \in C' \wedge (U \cap V \neq \{\})\}$

corollary *paracompact_closedin*:

fixes $S :: 'a :: \{\text{metric_space}, \text{second_countable_topology}\}$ set

assumes $cin: \text{closedin } (\text{top_of_set } U) S$

and $oin: \bigwedge T. T \in C \implies \text{openin } (\text{top_of_set } U) T$

and $S \subseteq \bigcup C$

obtains C' where $S \subseteq \bigcup C'$

and $\bigwedge V. V \in C' \implies \text{openin } (\text{top_of_set } U) V \wedge (\exists T. T \in C \wedge V \subseteq T)$

and $\bigwedge x. x \in U$

$\implies \exists V. \text{openin } (\text{top_of_set } U) V \wedge x \in V \wedge \text{finite } \{X. X \in C' \wedge (X \cap V \neq \{\})\}$

6.0.8 Covering an open set by a countable chain of compact sets

proposition *open_Union_compact_subsets*:

fixes $S :: 'a :: \text{euclidean_space}$ set

assumes $\text{open } S$

obtains C where $\bigwedge n. \text{compact } (C\ n) \wedge n. C\ n \subseteq S$

$\bigwedge n. C\ n \subseteq \text{interior}(C(\text{Suc } n))$

$\bigcup (\text{range } C) = S$

$\bigwedge K. [\text{compact } K; K \subseteq S] \implies \exists N. \forall n \geq N. K \subseteq (C\ n)$

6.0.9 Orthogonal complement

definition *orthogonal_comp* ($_^\perp$ [80] 80)

where $\text{orthogonal_comp } W \equiv \{x. \forall y \in W. \text{orthogonal } y\ x\}$

proposition *subspace_orthogonal_comp*: $\text{subspace } (W^\perp)$

proposition *subspace_sum_orthogonal_comp*:

fixes $U :: 'a :: \text{euclidean_space}$ set

assumes $\text{subspace } U$

shows $U + U^\perp = UNIV$

end

6.1 Path-Connectedness

theory *Path_Connected*

imports

Starlike

T1_Spaces

begin

6.1.1 Paths and Arcs

definition *path* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{bool}$

where $\text{path } g \equiv \text{continuous_on } \{0..1\} g$

definition *pathstart* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow 'a$

where $\text{pathstart } g \equiv g 0$

definition *pathfinish* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow 'a$

where $\text{pathfinish } g \equiv g 1$

definition *path_image* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow 'a \text{ set}$

where $\text{path_image } g \equiv g \text{ ` } \{0 .. 1\}$

definition *reversepath* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{real} \Rightarrow 'a$

where $\text{reversepath } g \equiv (\lambda x. g(1 - x))$

definition *joinpaths* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow (\text{real} \Rightarrow 'a) \Rightarrow \text{real} \Rightarrow 'a$
(infixr +++ 75)

where $g1 \text{ +++ } g2 \equiv (\lambda x. \text{if } x \leq 1/2 \text{ then } g1 (2 * x) \text{ else } g2 (2 * x - 1))$

definition *loop_free* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{bool}$

where $\text{loop_free } g \equiv \forall x \in \{0..1\}. \forall y \in \{0..1\}. g x = g y \longrightarrow x = y \vee x = 0 \wedge y = 1 \vee x = 1 \wedge y = 0$

definition *simple_path* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{bool}$

where $\text{simple_path } g \equiv \text{path } g \wedge \text{loop_free } g$

definition *arc* :: $(\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{bool}$

where $\text{arc } g \equiv \text{path } g \wedge \text{inj_on } g \{0..1\}$

6.1.2 Subpath

definition *subpath* :: $\text{real} \Rightarrow \text{real} \Rightarrow (\text{real} \Rightarrow 'a) \Rightarrow \text{real} \Rightarrow 'a::\text{real_normed_vector}$

where $\text{subpath } a b g \equiv \lambda x. g((b - a) * x + a)$

6.1.3 Shift Path to Start at Some Given Point

definition *shiftpath* :: $\text{real} \Rightarrow (\text{real} \Rightarrow 'a::\text{topological_space}) \Rightarrow \text{real} \Rightarrow 'a$

where $\text{shiftpath } a f = (\lambda x. \text{if } (a + x) \leq 1 \text{ then } f (a + x) \text{ else } f (a + x - 1))$

6.1.4 Straight-Line Paths

definition $linepath :: 'a::real_normed_vector \Rightarrow 'a \Rightarrow real \Rightarrow 'a$
 where $linepath\ a\ b = (\lambda x. (1 - x) *_{\mathbb{R}} a + x *_{\mathbb{R}} b)$

proposition $injective_eq_1d_open_map_UNIV:$

fixes $f :: real \Rightarrow real$

assumes $conf: continuous_on\ S\ f$ and $S: is_interval\ S$

shows $inj_on\ f\ S \longleftrightarrow (\forall T. open\ T \wedge T \subseteq S \longrightarrow open(f\ 'T))$
 (is ?lhs = ?rhs)

6.1.5 Path component

definition $path_component\ S\ x\ y \equiv$

$(\exists g. path\ g \wedge path_image\ g \subseteq S \wedge pathstart\ g = x \wedge pathfinish\ g = y)$

abbreviation

$path_component_set\ S\ x \equiv Collect\ (path_component\ S\ x)$

6.1.6 Path connectedness of a space

definition $path_connected\ S \longleftrightarrow$

$(\forall x \in S. \forall y \in S. \exists g. path\ g \wedge path_image\ g \subseteq S \wedge pathstart\ g = x \wedge pathfinish\ g = y)$

6.1.7 Path components

6.1.8 Paths and path-connectedness

6.1.9 Path components

6.1.10 Sphere is path-connected

corollary $connected_punctured_universe:$

$2 \leq DIM('N::euclidean_space) \implies connected(-\{a::'N\})$

proposition $path_connected_sphere:$

fixes $a :: 'a :: euclidean_space$

assumes $2 \leq DIM('a)$

shows $path_connected(sphere\ a\ r)$

corollary $path_connected_complement_bounded_convex:$

fixes $S :: 'a :: euclidean_space\ set$

assumes $bounded\ S\ convex\ S$ and $2: 2 \leq DIM('a)$

shows $\text{path_connected } (- S)$

proposition *connected_open_delete*:

assumes $\text{open } S$ $\text{connected } S$ **and** $2 \leq \text{DIM}('N::\text{euclidean_space})$
 shows $\text{connected}(S - \{a::'N\})$

corollary *path_connected_open_delete*:

assumes $\text{open } S$ $\text{connected } S$ **and** $2 \leq \text{DIM}('N::\text{euclidean_space})$
 shows $\text{path_connected}(S - \{a::'N\})$

corollary *path_connected_punctured_ball*:

$2 \leq \text{DIM}('N::\text{euclidean_space}) \implies \text{path_connected}(\text{ball } a \ r - \{a::'N\})$

corollary *connected_punctured_ball*:

$2 \leq \text{DIM}('N::\text{euclidean_space}) \implies \text{connected}(\text{ball } a \ r - \{a::'N\})$

corollary *connected_open_delete_finite*:

fixes $S \ T::'a::\text{euclidean_space set}$
 assumes S : $\text{open } S$ $\text{connected } S$ **and** $2 \leq \text{DIM}('a)$ **and** *finite* T
 shows $\text{connected}(S - T)$

6.1.11 Every annulus is a connected set

proposition *path_connected_annulus*:

fixes $a :: 'N::\text{euclidean_space}$
 assumes $2 \leq \text{DIM}('N)$
 shows $\text{path_connected } \{x. r1 < \text{norm}(x - a) \wedge \text{norm}(x - a) < r2\}$
 $\text{path_connected } \{x. r1 < \text{norm}(x - a) \wedge \text{norm}(x - a) \leq r2\}$
 $\text{path_connected } \{x. r1 \leq \text{norm}(x - a) \wedge \text{norm}(x - a) < r2\}$
 $\text{path_connected } \{x. r1 \leq \text{norm}(x - a) \wedge \text{norm}(x - a) \leq r2\}$

proposition *connected_annulus*:

fixes $a :: 'N::\text{euclidean_space}$
 assumes $2 \leq \text{DIM}('N::\text{euclidean_space})$
 shows $\text{connected } \{x. r1 < \text{norm}(x - a) \wedge \text{norm}(x - a) < r2\}$
 $\text{connected } \{x. r1 < \text{norm}(x - a) \wedge \text{norm}(x - a) \leq r2\}$
 $\text{connected } \{x. r1 \leq \text{norm}(x - a) \wedge \text{norm}(x - a) < r2\}$
 $\text{connected } \{x. r1 \leq \text{norm}(x - a) \wedge \text{norm}(x - a) \leq r2\}$

corollary *open_components*:

fixes $S :: 'a::\text{real_normed_vector set}$
 shows $[\text{open } u; S \in \text{components } u] \implies \text{open } S$

proposition *components_open_unique*:

fixes $S :: 'a::\text{real_normed_vector set}$
 assumes *pairwise disjoint* $A \cup A = S$
 $\bigwedge X. X \in A \implies \text{open } X \wedge \text{connected } X \wedge X \neq \{\}$

shows *components* $S = A$

6.1.12 The *inside* and *outside* of a Set

The *inside* comprises the points in a bounded connected component of the set's complement. The *outside* comprises the points in unbounded connected component of the complement.

definition *inside* **where**

$inside\ S \equiv \{x. (x \notin S) \wedge bounded(connect_component_set\ (-\ S)\ x)\}$

definition *outside* **where**

$outside\ S \equiv -S \cap \{x. \neg bounded(connect_component_set\ (-\ S)\ x)\}$

6.1.13 Condition for an open map's image to contain a ball

proposition *ball_subset_open_map_image:*

fixes $f :: 'a::heine_borel \Rightarrow 'b :: \{real_normed_vector, heine_borel\}$

assumes *contf*: $continuous_on\ (closure\ S)\ f$

and *oint*: $open\ (f\ 'interior\ S)$

and *le_no*: $\bigwedge z. z \in frontier\ S \implies r \leq norm(f\ z - f\ a)$

and *bounded* $S\ a \in S\ 0 < r$

shows $ball\ (f\ a)\ r \subseteq f\ 'S$

proposition *embedding_map_into_euclideanreal:*

assumes *path_connected_space* X

shows *embedding_map* $X\ euclideanreal\ f \longleftrightarrow$

$continuous_map\ X\ euclideanreal\ f \wedge inj_on\ f\ (topspace\ X)$

end

6.2 Neighbourhood bases and Locally path-connected spaces

theory *Locally*

imports

Path_Connected Function_Topology Sum_Topology

begin

6.2.1 Neighbourhood Bases

6.2.2 Locally path-connected spaces

6.2.3 Locally connected spaces

6.2.4 Dimension of a topological space

end

6.3 Some Uncountable Sets

```
theory Uncountable_Sets
  imports Path_Connected Continuum_Not_Denumerable
begin

end
```

6.4 Homotopy of Maps

```
theory Homotopy
  imports Path_Connected Product_Topology Uncountable_Sets
begin
```

definition *homotopic_with*

where

$$\begin{aligned} \text{homotopic_with } P \ X \ Y \ f \ g \equiv & \\ (\exists h. \text{continuous_map } (\text{prod_topology } (\text{top_of_set } \{0..1::\text{real}\}) \ X) \ Y \ h \wedge & \\ (\forall x. h(0, x) = f \ x) \wedge & \\ (\forall x. h(1, x) = g \ x) \wedge & \\ (\forall t \in \{0..1\}. P(\lambda x. h(t, x)))) & \end{aligned}$$

proposition *homotopic_with:*

assumes $\bigwedge h \ k. (\bigwedge x. x \in \text{topspace } X \implies h \ x = k \ x) \implies (P \ h \longleftrightarrow P \ k)$

shows $\text{homotopic_with } P \ X \ Y \ p \ q \longleftrightarrow$

$$\begin{aligned} & (\exists h. \text{continuous_map } (\text{prod_topology } (\text{subtopology euclideanreal } \{0..1\}) \\ & X) \ Y \ h \wedge \\ & (\forall x \in \text{topspace } X. h(0, x) = p \ x) \wedge \\ & (\forall x \in \text{topspace } X. h(1, x) = q \ x) \wedge \\ & (\forall t \in \{0..1\}. P(\lambda x. h(t, x)))) \end{aligned}$$

6.4.1 Homotopy with P is an equivalence relation

proposition *homotopic_with_trans:*

assumes $\text{homotopic_with } P \ X \ Y \ f \ g \ \text{homotopic_with } P \ X \ Y \ g \ h$

shows *homotopic_with* $P X Y f h$

6.4.2 Continuity lemmas

corollary *homotopic_compose*:

assumes *homotopic_with* $(\lambda x. True) X Y f f'$ *homotopic_with* $(\lambda x. True) Y Z g g'$
 shows *homotopic_with* $(\lambda x. True) X Z (g \circ f) (g' \circ f')$

proposition *homotopic_with_compose_continuous_right*:

\llbracket *homotopic_with_canon* $(\lambda f. p (f \circ h)) X Y f g$; *continuous_on* $W h$; $h \in W \rightarrow X$ \rrbracket
 \implies *homotopic_with_canon* $p W Y (f \circ h) (g \circ h)$

proposition *homotopic_with_compose_continuous_left*:

\llbracket *homotopic_with_canon* $(\lambda f. p (h \circ f)) X Y f g$; *continuous_on* $Y h$; $h \in Y \rightarrow Z$ \rrbracket
 \implies *homotopic_with_canon* $p X Z (h \circ f) (h \circ g)$

proposition *homotopic_with_eq*:

assumes h : *homotopic_with* $P X Y f g$
 and f' : $\bigwedge x. x \in \text{topspace } X \implies f' x = f x$
 and g' : $\bigwedge x. x \in \text{topspace } X \implies g' x = g x$
 and P : $(\bigwedge h k. (\bigwedge x. x \in \text{topspace } X \implies h x = k x) \implies P h \longleftrightarrow P k)$
 shows *homotopic_with* $P X Y f' g'$

6.4.3 Homotopy of paths, maintaining the same endpoints

definition *homotopic_paths* :: $['a \text{ set}, \text{real} \Rightarrow 'a, \text{real} \Rightarrow 'a::\text{topological_space}] \Rightarrow \text{bool}$

where

homotopic_paths $S p q \equiv$
homotopic_with_canon $(\lambda r. \text{pathstart } r = \text{pathstart } p \wedge \text{pathfinish } r = \text{pathfinish } p) \{0..1\} S p q$

proposition *homotopic_paths_imp_pathstart*:

homotopic_paths $S p q \implies \text{pathstart } p = \text{pathstart } q$

proposition *homotopic_paths_imp_pathfinish*:

homotopic_paths $S p q \implies \text{pathfinish } p = \text{pathfinish } q$

proposition *homotopic_paths_refl* [*simp*]: *homotopic_paths* $S p p \longleftrightarrow \text{path } p \wedge \text{path_image } p \subseteq S$

proposition *homotopic_paths_sym*: *homotopic_paths* $S p q \implies \text{homotopic_paths } S q p$

proposition *homotopic_paths_sym_eq*: $\text{homotopic_paths } S p q \longleftrightarrow \text{homotopic_paths } S q p$

proposition *homotopic_paths_trans* [trans]:
assumes *homotopic_paths* $S p q$ *homotopic_paths* $S q r$
shows *homotopic_paths* $S p r$

proposition *homotopic_paths_eq*:
 $\llbracket \text{path } p; \text{path_image } p \subseteq S; \bigwedge t. t \in \{0..1\} \implies p t = q t \rrbracket \implies \text{homotopic_paths } S p q$

proposition *homotopic_paths_reparametrize*:
assumes *path* p
and *pips*: $\text{path_image } p \subseteq S$
and *contf*: *continuous_on* $\{0..1\} f$
and *f01* : $f \in \{0..1\} \rightarrow \{0..1\}$
and [*simp*]: $f(0) = 0 \ f(1) = 1$
and *q*: $\bigwedge t. t \in \{0..1\} \implies q(t) = p(f t)$
shows *homotopic_paths* $S p q$

proposition *homotopic_paths_reversepath*:
 $\text{homotopic_paths } S (\text{reversepath } p) (\text{reversepath } q) \longleftrightarrow \text{homotopic_paths } S p q$

proposition *homotopic_paths_join*:
 $\llbracket \text{homotopic_paths } S p p'; \text{homotopic_paths } S q q'; \text{pathfinish } p = \text{pathstart } q \rrbracket \implies \text{homotopic_paths } S (p +++ q) (p' +++ q')$

proposition *homotopic_paths_continuous_image*:
 $\llbracket \text{homotopic_paths } S f g; \text{continuous_on } S h; h \in S \rightarrow t \rrbracket \implies \text{homotopic_paths } t (h \circ f) (h \circ g)$

6.4.4 Group properties for homotopy of paths

So taking equivalence classes under homotopy would give the fundamental group

proposition *homotopic_paths_rid*:
assumes *path* p $\text{path_image } p \subseteq S$
shows *homotopic_paths* $S (p +++ \text{linepath } (\text{pathfinish } p) (\text{pathfinish } p)) p$

proposition *homotopic_paths_lid*:
 $\llbracket \text{path } p; \text{path_image } p \subseteq S \rrbracket \implies \text{homotopic_paths } S (\text{linepath } (\text{pathstart } p) (\text{pathstart } p) +++ p) p$

proposition *homotopic_paths_assoc*:
 $\llbracket \text{path } p; \text{path_image } p \subseteq S; \text{path } q; \text{path_image } q \subseteq S; \text{path } r; \text{path_image } r \subseteq S \rrbracket$

S ; $\text{pathfinish } p = \text{pathstart } q$;
 $\text{pathfinish } q = \text{pathstart } r$]]
 $\implies \text{homotopic_paths } S (p \text{ +++ } (q \text{ +++ } r)) ((p \text{ +++ } q) \text{ +++ } r)$

proposition *homotopic_paths_rinv*:
assumes $\text{path } p \text{ path_image } p \subseteq S$
shows $\text{homotopic_paths } S (p \text{ +++ } \text{reversepath } p) (\text{linepath } (\text{pathstart } p) (\text{pathstart } p))$

proposition *homotopic_paths_linv*:
assumes $\text{path } p \text{ path_image } p \subseteq S$
shows $\text{homotopic_paths } S (\text{reversepath } p \text{ +++ } p) (\text{linepath } (\text{pathfinish } p) (\text{pathfinish } p))$

6.4.5 Homotopy of loops without requiring preservation of endpoints

definition *homotopic_loops* :: $'a::\text{topological_space } \text{set} \Rightarrow (\text{real} \Rightarrow 'a) \Rightarrow (\text{real} \Rightarrow 'a) \Rightarrow \text{bool}$ **where**
 $\text{homotopic_loops } S p q \equiv$
 $\text{homotopic_with_canon } (\lambda r. \text{pathfinish } r = \text{pathstart } r) \{0..1\} S p q$

proposition *homotopic_loops_imp_loop*:
 $\text{homotopic_loops } S p q \implies \text{pathfinish } p = \text{pathstart } p \wedge \text{pathfinish } q = \text{pathstart } q$

proposition *homotopic_loops_imp_path*:
 $\text{homotopic_loops } S p q \implies \text{path } p \wedge \text{path } q$

proposition *homotopic_loops_imp_subset*:
 $\text{homotopic_loops } S p q \implies \text{path_image } p \subseteq S \wedge \text{path_image } q \subseteq S$

proposition *homotopic_loops_refl*:
 $\text{homotopic_loops } S p p \longleftrightarrow$
 $\text{path } p \wedge \text{path_image } p \subseteq S \wedge \text{pathfinish } p = \text{pathstart } p$

proposition *homotopic_loops_sym*: $\text{homotopic_loops } S p q \implies \text{homotopic_loops } S q p$

proposition *homotopic_loops_sym_eq*: $\text{homotopic_loops } S p q \longleftrightarrow \text{homotopic_loops } S q p$

proposition *homotopic_loops_trans*:
 $[[\text{homotopic_loops } S p q; \text{homotopic_loops } S q r]] \implies \text{homotopic_loops } S p r$

proposition *homotopic_loops_subset*:
 $[[\text{homotopic_loops } S p q; S \subseteq t]] \implies \text{homotopic_loops } t p q$

proposition *homotopic_loops_eq*:

$\llbracket \text{path } p; \text{path_image } p \subseteq S; \text{pathfinish } p = \text{pathstart } p; \bigwedge t. t \in \{0..1\} \implies p(t) = q(t) \rrbracket$
 $\implies \text{homotopic_loops } S \ p \ q$

proposition *homotopic_loops_continuous_image*:

$\llbracket \text{homotopic_loops } S \ f \ g; \text{continuous_on } S \ h; h \in S \rightarrow t \rrbracket \implies \text{homotopic_loops } t \ (h \circ f) \ (h \circ g)$

6.4.6 Relations between the two variants of homotopy

proposition *homotopic_paths_imp_homotopic_loops*:

$\llbracket \text{homotopic_paths } S \ p \ q; \text{pathfinish } p = \text{pathstart } p; \text{pathfinish } q = \text{pathstart } p \rrbracket$
 $\implies \text{homotopic_loops } S \ p \ q$

proposition *homotopic_loops_imp_homotopic_paths_null*:

assumes *homotopic_loops* $S \ p \ (\text{linepath } a \ a)$
shows *homotopic_paths* $S \ p \ (\text{linepath } (\text{pathstart } p) \ (\text{pathstart } p))$

proposition *homotopic_loops_conjugate*:

fixes $S :: 'a::\text{real_normed_vector_set}$
assumes *path* $p \ \text{path } q$ **and** *pip*: $\text{path_image } p \subseteq S$ **and** *piq*: $\text{path_image } q \subseteq S$
and *pq*: $\text{pathfinish } p = \text{pathstart } q$ **and** *qloop*: $\text{pathfinish } q = \text{pathstart } q$
shows *homotopic_loops* $S \ (p \ +++ \ q \ +++ \ \text{reversepath } p) \ q$

6.4.7 Homotopy and subpaths

proposition *homotopic_join_subpaths*:

$\llbracket \text{path } g; \text{path_image } g \subseteq S; u \in \{0..1\}; v \in \{0..1\}; w \in \{0..1\} \rrbracket$
 $\implies \text{homotopic_paths } S \ (\text{subpath } u \ v \ g \ +++ \ \text{subpath } v \ w \ g) \ (\text{subpath } u \ w \ g)$

6.4.8 Simply connected sets

defined as "all loops are homotopic (as loops)"

definition *simply_connected* **where**

simply_connected $S \equiv$
 $\forall p \ q. \text{path } p \wedge \text{pathfinish } p = \text{pathstart } p \wedge \text{path_image } p \subseteq S \wedge$
 $\text{path } q \wedge \text{pathfinish } q = \text{pathstart } q \wedge \text{path_image } q \subseteq S$
 $\longrightarrow \text{homotopic_loops } S \ p \ q$

proposition *simply_connected_Times*:

fixes $S :: 'a::\text{real_normed_vector_set}$ **and** $T :: 'b::\text{real_normed_vector_set}$
assumes S : *simply_connected* S **and** T : *simply_connected* T
shows *simply_connected* $(S \times T)$

6.4.9 Contractible sets

definition *contractible where*

contractible $S \equiv \exists a. \text{homotopic_with_canon } (\lambda x. \text{True}) S S \text{id } (\lambda x. a)$

proposition *contractible imp simply connected:*

fixes $S :: _ :: \text{real_normed_vector_set}$

assumes *contractible* S **shows** *simply connected* S

corollary *contractible imp connected:*

fixes $S :: _ :: \text{real_normed_vector_set}$

shows *contractible* $S \implies$ *connected* S

6.4.10 Starlike sets

definition *starlike* $S \longleftrightarrow (\exists a \in S. \forall x \in S. \text{closed_segment } a x \subseteq S)$

6.4.11 Local versions of topological properties in general

definition *locally* $:: ('a :: \text{topological_space } \text{set} \Rightarrow \text{bool}) \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$

where

locally $P S \equiv$

$\forall w x. \text{openin } (\text{top_of_set } S) w \wedge x \in w$

$\longrightarrow (\exists U V. \text{openin } (\text{top_of_set } S) U \wedge P V \wedge x \in U \wedge U \subseteq V \wedge V$

$\subseteq w)$

proposition *homeomorphism locally imp:*

fixes $S :: 'a :: \text{metric_space } \text{set}$ **and** $T :: 'b :: \text{t2_space } \text{set}$

assumes S : *locally* $P S$ **and** hom : *homeomorphism* $S T f g$

and Q : $\bigwedge S S'. \llbracket P S; \text{homeomorphism } S S' f g \rrbracket \implies Q S'$

shows *locally* $Q T$

6.4.12 An induction principle for connected sets

proposition *connected induction:*

assumes *connected* S

and opD : $\bigwedge T a. \llbracket \text{openin } (\text{top_of_set } S) T; a \in T \rrbracket \implies \exists z. z \in T \wedge P z$

and opI : $\bigwedge a. a \in S$

$\implies \exists T. \text{openin } (\text{top_of_set } S) T \wedge a \in T \wedge$

$(\forall x \in T. \forall y \in T. P x \wedge P y \wedge Q x \longrightarrow Q y)$

and *etc*: $a \in S b \in S P a P b Q a$

shows $Q b$

6.4.13 Basic properties of local compactness

proposition *locally_compact:*

fixes $S :: 'a :: \text{metric_space set}$

shows

$\text{locally_compact } S \longleftrightarrow$
 $(\forall x \in S. \exists u v. x \in u \wedge u \subseteq v \wedge v \subseteq S \wedge$
 $\quad \text{openin } (\text{top_of_set } S) u \wedge \text{compact } v)$
(is ?lhs = ?rhs)

6.4.14 Sura-Bura's results about compact components of sets

proposition *Sura_Bura_compact:*

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $\text{compact } S$ **and** $C: C \in \text{components } S$

shows $C = \bigcap \{T. C \subseteq T \wedge \text{openin } (\text{top_of_set } S) T \wedge$
 $\quad \text{closedin } (\text{top_of_set } S) T\}$

(is $C = \bigcap ?\mathcal{T}$)

corollary *Sura_Bura_clopen_subset:*

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $S: \text{locally_compact } S$ **and** $C: C \in \text{components } S$ **and** $\text{compact } C$

and $U: \text{open } U \ C \subseteq U$

obtains K **where** $\text{openin } (\text{top_of_set } S) K$ $\text{compact } K \ C \subseteq K \ K \subseteq U$

corollary *Sura_Bura_clopen_subset_alt:*

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $S: \text{locally_compact } S$ **and** $C: C \in \text{components } S$ **and** $\text{compact } C$

and $\text{open } U: \text{openin } (\text{top_of_set } S) U$ **and** $C \subseteq U$

obtains K **where** $\text{openin } (\text{top_of_set } S) K$ $\text{compact } K \ C \subseteq K \ K \subseteq U$

corollary *Sura_Bura:*

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $\text{locally_compact } S \ C \in \text{components } S \ \text{compact } C$

shows $C = \bigcap \{K. C \subseteq K \wedge \text{compact } K \wedge \text{openin } (\text{top_of_set } S) K\}$

(is $C = ?rhs$)

6.4.15 Special cases of local connectedness and path connectedness

proposition *locally_path_connected:*

$\text{locally_path_connected } S \longleftrightarrow$

$(\forall V x. \text{openin } (\text{top_of_set } S) V \wedge x \in V$

$\longrightarrow (\exists U. \text{openin } (\text{top_of_set } S) U \wedge \text{path_connected } U \wedge x \in U \wedge U \subseteq$

$V))$

proposition *locally_path_connected_open_path_component:*

$$\begin{aligned} & \text{locally_path_connected } S \longleftrightarrow \\ & (\forall t x. \text{openin } (\text{top_of_set } S) t \wedge x \in t \\ & \quad \longrightarrow \text{openin } (\text{top_of_set } S) (\text{path_component_set } t x)) \end{aligned}$$

proposition *locally_connected_im_kleinen:*

$$\begin{aligned} & \text{locally_connected } S \longleftrightarrow \\ & (\forall v x. \text{openin } (\text{top_of_set } S) v \wedge x \in v \\ & \quad \longrightarrow (\exists u. \text{openin } (\text{top_of_set } S) u \wedge \\ & \quad \quad x \in u \wedge u \subseteq v \wedge \\ & \quad \quad (\forall y. y \in u \longrightarrow (\exists c. \text{connected } c \wedge c \subseteq v \wedge x \in c \wedge y \in c)))) \\ & \text{(is ?lhs = ?rhs)} \end{aligned}$$

proposition *locally_path_connected_im_kleinen:*

$$\begin{aligned} & \text{locally_path_connected } S \longleftrightarrow \\ & (\forall v x. \text{openin } (\text{top_of_set } S) v \wedge x \in v \\ & \quad \longrightarrow (\exists u. \text{openin } (\text{top_of_set } S) u \wedge \\ & \quad \quad x \in u \wedge u \subseteq v \wedge \\ & \quad \quad (\forall y. y \in u \longrightarrow (\exists p. \text{path } p \wedge \text{path_image } p \subseteq v \wedge \\ & \quad \quad \quad \text{pathstart } p = x \wedge \text{pathfinish } p = y)))) \\ & \text{(is ?lhs = ?rhs)} \end{aligned}$$

6.4.16 Relations between components and path components

proposition *locally_connected_quotient_image:*

$$\begin{aligned} & \text{assumes } \text{lcS}: \text{locally_connected } S \\ & \text{and } \text{oo}: \bigwedge T. T \subseteq f \text{' } S \\ & \quad \Longrightarrow \text{openin } (\text{top_of_set } S) (S \cap f \text{' } T) \longleftrightarrow \\ & \quad \quad \text{openin } (\text{top_of_set } (f \text{' } S)) T \\ & \text{shows } \text{locally_connected } (f \text{' } S) \end{aligned}$$

proposition *locally_path_connected_quotient_image:*

$$\begin{aligned} & \text{assumes } \text{lcS}: \text{locally_path_connected } S \\ & \text{and } \text{oo}: \bigwedge T. T \subseteq f \text{' } S \\ & \quad \Longrightarrow \text{openin } (\text{top_of_set } S) (S \cap f \text{' } T) \longleftrightarrow \text{openin } (\text{top_of_set } (f \\ & \quad \text{' } S)) T \\ & \text{shows } \text{locally_path_connected } (f \text{' } S) \end{aligned}$$

6.4.17 Existence of isometry between subspaces of same dimension

proposition *isometries_subspaces:*

$$\begin{aligned} & \text{fixes } S :: 'a::\text{euclidean_space set} \\ & \text{and } T :: 'b::\text{euclidean_space set} \\ & \text{assumes } S: \text{subspace } S \end{aligned}$$

```

and  $T$ : subspace  $T$ 
and  $d$ :  $\dim S = \dim T$ 
obtains  $f g$  where linear  $f$  linear  $g$   $f \text{ ' } S = T$   $g \text{ ' } T = S$ 
 $\bigwedge x. x \in S \implies \text{norm}(f x) = \text{norm } x$ 
 $\bigwedge x. x \in T \implies \text{norm}(g x) = \text{norm } x$ 
 $\bigwedge x. x \in S \implies g(f x) = x$ 
 $\bigwedge x. x \in T \implies f(g x) = x$ 

```

corollary *isometry_subspaces*:

```

fixes  $S :: 'a::\text{euclidean\_space set}$ 
and  $T :: 'b::\text{euclidean\_space set}$ 
assumes  $S$ : subspace  $S$ 
and  $T$ : subspace  $T$ 
and  $d$ :  $\dim S = \dim T$ 
obtains  $f$  where linear  $f$   $f \text{ ' } S = T$   $\bigwedge x. x \in S \implies \text{norm}(f x) = \text{norm } x$ 

```

corollary *isomorphisms_UNIV_UNIV*:

```

assumes  $\text{DIM}('M) = \text{DIM}('N)$ 
obtains  $f :: 'M::\text{euclidean\_space} \Rightarrow 'N::\text{euclidean\_space}$  and  $g$ 
where linear  $f$  linear  $g$ 
 $\bigwedge x. \text{norm}(f x) = \text{norm } x$   $\bigwedge y. \text{norm}(g y) = \text{norm } y$ 
 $\bigwedge x. g (f x) = x$   $\bigwedge y. f(g y) = y$ 

```

6.4.18 Retracts, in a general sense, preserve (co)homotopic triviality)

locale *Retracts* =

```

fixes  $S h t k$ 
assumes  $\text{conth}$ : continuous_on  $S h$ 
and  $\text{imh}$ :  $h \text{ ' } S = t$ 
and  $\text{contk}$ : continuous_on  $t k$ 
and  $\text{imk}$ :  $k \in t \rightarrow S$ 
and  $\text{idhk}$ :  $\bigwedge y. y \in t \implies h(k y) = y$ 

```

begin

6.4.19 Homotopy equivalence

6.4.20 Homotopy equivalence of topological spaces.

definition *homotopy_equivalent_space*

(**infix** *homotopy'_equivalent'_space* 50)

```

where  $X$  homotopy_equivalent_space  $Y \equiv$ 
 $(\exists f g. \text{continuous\_map } X Y f \wedge$ 
 $\text{continuous\_map } Y X g \wedge$ 
 $\text{homotopic\_with } (\lambda x. \text{True}) X X (g \circ f) \text{ id} \wedge$ 
 $\text{homotopic\_with } (\lambda x. \text{True}) Y Y (f \circ g) \text{ id})$ 

```

6.4.21 Contractible spaces

corollary *contractible_space_euclideanreal*: *contractible_space euclideanreal*

abbreviation *homotopy_eqv* :: '*a*::*topological_space set* \Rightarrow '*b*::*topological_space set* \Rightarrow *bool*

(**infix** *homotopy'_eqv* 50)

where *S homotopy_eqv T* \equiv *top_of_set S homotopy_equivalent_space top_of_set T*

corollary *bounded_path_connected_CompL_real*:

fixes *S* :: *real set*

assumes *bounded S path_connected(- S)* **shows** *S = {}*

proposition *path_connected_convex_diff_countable*:

fixes *U* :: '*a*::*euclidean_space set*

assumes *convex U \neg collinear U countable S*

shows *path_connected(U - S)*

corollary *connected_convex_diff_countable*:

fixes *U* :: '*a*::*euclidean_space set*

assumes *convex U \neg collinear U countable S*

shows *connected(U - S)*

proposition *path_connected_openin_diff_countable*:

fixes *S* :: '*a*::*euclidean_space set*

assumes *connected S and ope: openin (top_of_set (affine hull S)) S*

and \neg *collinear S countable T*

shows *path_connected(S - T)*

corollary *connected_openin_diff_countable*:

fixes *S* :: '*a*::*euclidean_space set*

assumes *connected S and ope: openin (top_of_set (affine hull S)) S*

and \neg *collinear S countable T*

shows *connected(S - T)*

corollary *path_connected_open_diff_countable*:

fixes *S* :: '*a*::*euclidean_space set*

assumes $2 \leq DIM('a)$ *open S connected S countable T*

shows *path_connected(S - T)*

corollary *connected_open_diff_countable*:

fixes *S* :: '*a*::*euclidean_space set*

assumes $2 \leq \text{DIM}(a)$ *open S connected S countable T*
shows $\text{connected}(S - T)$

6.4.22 Nullhomotopic mappings

proposition *nullhomotopic_from_sphere_extension:*

fixes $f :: 'M::\text{euclidean_space} \Rightarrow 'a::\text{real_normed_vector}$

shows $(\exists c. \text{homotopic_with_canon } (\lambda x. \text{True}) (\text{sphere } a \ r) \ S \ f \ (\lambda x. c)) \longleftrightarrow$

$(\exists g. \text{continuous_on } (\text{cball } a \ r) \ g \wedge g \ ' (\text{cball } a \ r) \subseteq S \wedge$

$(\forall x \in \text{sphere } a \ r. g \ x = f \ x))$

(is ?lhs = ?rhs)

end

6.5 Euclidean space and n-spheres, as subtopologies of n-dimensional space

theory *Abstract_Euclidean_Space*

imports *Homotopy Locally*

begin

6.5.1 Euclidean spaces as abstract topologies

6.5.2 n-dimensional spheres

proposition *contractible_space_upper_hemisphere:*

assumes $k \leq n$

shows $\text{contractible_space}(\text{subtopology } (\text{nsphere } n) \ \{x. x \ k \geq 0\})$

corollary *contractible_space_lower_hemisphere:*

assumes $k \leq n$

shows $\text{contractible_space}(\text{subtopology } (\text{nsphere } n) \ \{x. x \ k \leq 0\})$

proposition *nullhomotopic_nonsurjective_sphere_map:*

assumes $f: \text{continuous_map } (\text{nsphere } p) \ (\text{nsphere } p) \ f$

and $\text{fim: } f \ ' (\text{topspace}(\text{nsphere } p)) \neq \text{topspace}(\text{nsphere } p)$

obtains a **where** $\text{homotopic_with } (\lambda x. \text{True}) (\text{nsphere } p) \ (\text{nsphere } p) \ f \ (\lambda x. a)$

end

6.6 Various Forms of Topological Spaces

theory *Abstract_Topological_Spaces*
imports *Lindelof_Spaces Locally_Abstract_Euclidean_Space Sum_Topology FSigma*
begin

6.6.1 Connected topological spaces

6.6.2 The notion of "separated between" (complement of "connected between")

6.6.3 Connected components

6.6.4 Monotone maps (in the general topological sense)

proposition *connected_space_monotone_quotient_map_preimage:*
assumes *f: monotone_map X Y f quotient_map X Y f* **and** *connected_space Y*
shows *connected_space X*

6.6.5 Other countability properties

6.6.6 Neighbourhood bases EXTRAS

6.6.7 T_0 spaces and the Kolmogorov quotient

proposition *t0_space_product_topology:*
 $t0_space (product_topology X I) \iff product_topology X I = trivial_topology$
 $\vee (\forall i \in I. t0_space (X i))$
(is ?lhs=?rhs)

6.6.8 Kolmogorov quotients

6.6.9 Closed diagonals and graphs

6.6.10 KC spaces, those where all compact sets are closed.

proposition *kc_space_prod_topology_left:*
assumes X : *kc_space* X **and** Y : *Hausdorff_space* Y
shows *kc_space* (*prod_topology* X Y)

6.6.11 Technical results about proper maps, perfect maps, etc

6.6.12 Regular spaces

proposition *regular_space_continuous_proper_map_image:*
assumes *regular_space* X **and** *contf*: *continuous_map* X Y f **and** *pmapf*:
proper_map X Y f
and *fm*: $f^{-1}(\text{topspace } Y) = \text{topspace } X$
shows *regular_space* Y

proposition *regular_space_perfect_map_image_eq:*
assumes *Hausdorff_space* X **and** *perf*: *perfect_map* X Y f
shows *regular_space* $X \longleftrightarrow$ *regular_space* Y (**is** ?lhs=?rhs)

6.6.13 Locally compact spaces

proposition *quotient_map_prod_right:*
assumes *loc*: *locally_compact_space* Z
and *reg*: *Hausdorff_space* $Z \vee$ *regular_space* Z
and *f*: *quotient_map* X Y f
shows *quotient_map* (*prod_topology* Z X) (*prod_topology* Z Y) $(\lambda(x,y). (x,f y))$

6.6.14 Special characterizations of classes of functions into and out of \mathbb{R}

6.6.15 Normal spaces

6.6.16 Hereditary topological properties

6.6.17 Limits in a topological space

6.6.18 Quasi-components

6.6.19 Additional quasicomponent and continuum properties like Boundary Bumping

6.6.20 Compactly generated spaces (k-spaces)

end

6.7 Abstract Metric Spaces

theory *Abstract_Metric_Spaces*

imports *Elementary_Metric_Spaces Abstract_Limits Abstract_Topological_Spaces*

begin

6.7.1 Metric topology

6.7.2 Bounded sets

6.7.3 Subspace of a metric space

6.7.4 Abstract type of metric spaces

6.7.5 The discrete metric

6.7.6 Metrizable spaces

6.7.7 Limits at a point in a topological space

6.7.8 Normal spaces and metric spaces

6.7.9 Topological limit in metric spaces

6.7.10 Cauchy sequences and complete metric spaces

6.7.11 Totally bounded subsets of metric spaces

6.7.12 Compactness in metric spaces

6.7.13 Continuous functions on metric spaces**6.7.14** Completely metrizable spaces**6.7.15** Product metric**6.7.16** More sequential characterizations in a metric space**6.7.17** Three strong notions of continuity for metric spaces**6.7.18** Isometries**6.7.19** "Capped" equivalent bounded metrics and general product metrics

proposition *metrizable_space_product_topology:*
 $\text{metrizable_space } (\text{product_topology } X I) \longleftrightarrow$
 $(\text{product_topology } X I) = \text{trivial_topology} \vee$

countable $\{i \in I. \neg (\exists a. \text{topspace}(X\ i) \subseteq \{a\})\} \wedge$
 $(\forall i \in I. \text{metrizable_space } (X\ i))$

proposition *completely_metrizable_space_product_topology:*
 $\text{completely_metrizable_space } (\text{product_topology } X\ I) \longleftrightarrow$
 $(\text{product_topology } X\ I) = \text{trivial_topology} \vee$
countable $\{i \in I. \neg (\exists a. \text{topspace}(X\ i) \subseteq \{a\})\} \wedge$
 $(\forall i \in I. \text{completely_metrizable_space } (X\ i))$

end

6.8 Infinite sums

theory *Infinite_Sum*

imports

Elementary_Topology

HOL-Library.Extended_Nonnegative_Real

HOL-Library.Complex_Order

begin

6.8.1 Definition and syntax

6.8.2 General properties

6.8.3 Absolute convergence

6.8.4 Extended reals and nats

6.8.5 Real numbers

6.8.6 Complex numbers

```

class complete_uniform_space = uniform_space +
  assumes cauchy_filter_convergent': cauchy_filter (F :: 'a filter)  $\implies$  F  $\neq$  bot
 $\implies$  convergent_filter F

```

```

theorem (in uniform_space) controlled_sequences_convergent_imp_complete:
  fixes U :: nat  $\Rightarrow$  ('a  $\times$  'a) set
  assumes gen: countably_generated_filter (uniformity :: ('a  $\times$  'a) filter)
  assumes U:  $\bigwedge n$ . eventually ( $\lambda z$ . z  $\in$  U n) uniformity
  assumes conv:  $\bigwedge (u :: \text{nat} \Rightarrow 'a)$ . ( $\bigwedge N m n$ . N  $\leq$  m  $\implies$  N  $\leq$  n  $\implies$  (u m, u n)
 $\in$  U N)  $\implies$  convergent u
  shows class.complete_uniform_space open uniformity

```

```

theorem (in uniform_space) controlled_seq_imp_Cauchy_seq:
  fixes U :: nat  $\Rightarrow$  ('a  $\times$  'a) set
  assumes U:  $\bigwedge P$ . eventually P uniformity  $\implies$  ( $\exists n$ .  $\forall x \in U n$ . P x)
  assumes controlled:  $\bigwedge N m n$ . N  $\leq$  m  $\implies$  N  $\leq$  n  $\implies$  (f m, f n)  $\in$  U N
  shows Cauchy f

```

```

theorem (in uniform_space) Cauchy_seq_convergent_imp_complete:
  fixes U :: nat  $\Rightarrow$  ('a  $\times$  'a) set
  assumes gen: countably_generated_filter (uniformity :: ('a  $\times$  'a) filter)
  assumes conv:  $\bigwedge (u :: \text{nat} \Rightarrow 'a)$ . Cauchy u  $\implies$  convergent u
  shows class.complete_uniform_space open uniformity

```

end

6.9 Ordered Euclidean Space

```

theory Ordered_Euclidean_Space
imports
  Convex_Euclidean_Space Abstract_Limits
  HOL-Library.Product_Order
beginclass ordered_euclidean_space = ord + inf + sup + abs + Inf + Sup +
euclidean_space +

```

```

assumes eucl_le:  $x \leq y \iff (\forall i \in \text{Basis}. x \cdot i \leq y \cdot i)$ 
assumes eucl_less_le_not_le:  $x < y \iff x \leq y \wedge \neg y \leq x$ 
assumes eucl_inf:  $\text{inf } x \ y = (\sum_{i \in \text{Basis}. \text{inf } (x \cdot i) (y \cdot i) *_{\mathbb{R}} i)$ 
assumes eucl_sup:  $\text{sup } x \ y = (\sum_{i \in \text{Basis}. \text{sup } (x \cdot i) (y \cdot i) *_{\mathbb{R}} i)$ 
assumes eucl_Inf:  $\text{Inf } X = (\sum_{i \in \text{Basis}. (\text{INF } x \in X. x \cdot i) *_{\mathbb{R}} i)$ 
assumes eucl_Sup:  $\text{Sup } X = (\sum_{i \in \text{Basis}. (\text{SUP } x \in X. x \cdot i) *_{\mathbb{R}} i)$ 
assumes eucl_abs:  $|x| = (\sum_{i \in \text{Basis}. |x \cdot i| *_{\mathbb{R}} i)$ 
begin

proposition compact_attains_Inf_componentwise:
  fixes b::'a::ordered_euclidean_space
  assumes b  $\in$  Basis assumes X  $\neq$  {} compact X
  obtains x where  $x \in X \ x \cdot b = \text{Inf } X \cdot b \ \wedge \ y. y \in X \implies x \cdot b \leq y \cdot b$ 

proposition
  compact_attains_Sup_componentwise:
  fixes b::'a::ordered_euclidean_space
  assumes b  $\in$  Basis assumes X  $\neq$  {} compact X
  obtains x where  $x \in X \ x \cdot b = \text{Sup } X \cdot b \ \wedge \ y. y \in X \implies y \cdot b \leq x \cdot b$ 

proposition
  fixes a :: 'a::ordered_euclidean_space
  shows cbox_interval:  $\text{cbox } a \ b = \{a..b\}$ 
  and interval_cbox:  $\{a..b\} = \text{cbox } a \ b$ 
  and eucl_le_atMost:  $\{x. \forall i \in \text{Basis}. x \cdot i \leq a \cdot i\} = \{..a\}$ 
  and eucl_le_atLeast:  $\{x. \forall i \in \text{Basis}. a \cdot i \leq x \cdot i\} = \{a..\}$ 

instantiation vec :: (ordered_euclidean_space, finite) ordered_euclidean_space
begin

definition inf x y = ( $\chi$  i. inf (x $ i) (y $ i))
definition sup x y = ( $\chi$  i. sup (x $ i) (y $ i))
definition Inf X = ( $\chi$  i. (INF x  $\in$  X. x $ i))
definition Sup X = ( $\chi$  i. (SUP x  $\in$  X. x $ i))
definition |x| = ( $\chi$  i. |x $ i|)

end

```

6.10 Arcwise-Connected Sets

```

theory Arcwise_Connected
imports Path_Connected Ordered_Euclidean_Space HOL-Computational_Algebra.Primes
begin

```

6.10.1 The Brouwer reduction theorem

```

theorem Brouwer_reduction_theorem_gen:
  fixes S :: 'a::euclidean_space set
  assumes closed S  $\varphi$  S

```

and $\varphi: \bigwedge F. \llbracket \bigwedge n. \text{closed}(F\ n); \bigwedge n. \varphi(F\ n); \bigwedge n. F(\text{Suc}\ n) \subseteq F\ n \rrbracket \implies \varphi(\bigcap(\text{range}\ F))$
obtains T **where** $T \subseteq S$ $\text{closed}\ T$ $\varphi\ T$ $\bigwedge U. \llbracket U \subseteq S; \text{closed}\ U; \varphi\ U \rrbracket \implies \neg(U \subset T)$

corollary *Brouwer_reduction_theorem*:

fixes $S :: 'a::\text{euclidean_space}\ \text{set}$
assumes $\text{compact}\ S$ $\varphi\ S$ $S \neq \{\}$
and $\varphi: \bigwedge F. \llbracket \bigwedge n. \text{compact}(F\ n); \bigwedge n. F\ n \neq \{\}; \bigwedge n. \varphi(F\ n); \bigwedge n. F(\text{Suc}\ n) \subseteq F\ n \rrbracket \implies \varphi(\bigcap(\text{range}\ F))$
obtains T **where** $T \subseteq S$ $\text{compact}\ T$ $T \neq \{\}$ $\varphi\ T$
 $\bigwedge U. \llbracket U \subseteq S; \text{closed}\ U; U \neq \{\}; \varphi\ U \rrbracket \implies \neg(U \subset T)$

6.10.2 Density of points with dyadic rational coordinates

proposition *closure_dyadic_rationals*:

$\text{closure}(\bigcup k. \bigcup f \in \text{Basis} \rightarrow \mathbb{Z}. \{ \sum i :: 'a :: \text{euclidean_space} \in \text{Basis}. (f\ i / 2^k) *_R i \}) = \text{UNIV}$

corollary *closure_rational_coordinates*:

$\text{closure}(\bigcup f \in \text{Basis} \rightarrow \mathbb{Q}. \{ \sum i :: 'a :: \text{euclidean_space} \in \text{Basis}. f\ i *_R i \}) = \text{UNIV}$

theorem *homeomorphic_monotone_image_interval*:

fixes $f :: \text{real} \Rightarrow 'a::\{\text{real_normed_vector},\text{complete_space}\}$
assumes $\text{cont}_f: \text{continuous_on}\ \{0..1\}\ f$
and $\text{conn}: \bigwedge y. \text{connected}\ (\{0..1\} \cap f^{-1}\ \{y\})$
and $f_1\text{not}0: f\ 1 \neq f\ 0$
shows $(f\ ^\ \{0..1\})$ *homeomorphic* $\{0..1::\text{real}\}$

theorem *path_contains_arc*:

fixes $p :: \text{real} \Rightarrow 'a::\{\text{complete_space},\text{real_normed_vector}\}$
assumes $\text{path}\ p$ **and** $a: \text{pathstart}\ p = a$ **and** $b: \text{pathfinish}\ p = b$ **and** $a \neq b$
obtains q **where** $\text{arc}\ q$ $\text{path_image}\ q \subseteq \text{path_image}\ p$ $\text{pathstart}\ q = a$ $\text{pathfinish}\ q = b$

corollary *path_connected_arcwise*:

fixes $S :: 'a::\{\text{complete_space},\text{real_normed_vector}\}\ \text{set}$
shows $\text{path_connected}\ S \longleftrightarrow$

$$(\forall x \in S. \forall y \in S. x \neq y \longrightarrow (\exists g. \text{arc } g \wedge \text{path_image } g \subseteq S \wedge \text{pathstart } g = x \wedge \text{pathfinish } g = y))$$

(is ?lhs = ?rhs)

corollary *arc_connected_trans*:

fixes $g :: \text{real} \Rightarrow 'a::\{\text{complete_space}, \text{real_normed_vector}\}$
assumes $\text{arc } g \text{ arc } h \text{ pathfinish } g = \text{pathstart } h \text{ pathstart } g \neq \text{pathfinish } h$
obtains i **where** $\text{arc } i \text{ path_image } i \subseteq \text{path_image } g \cup \text{path_image } h$
 $\text{pathstart } i = \text{pathstart } g \text{ pathfinish } i = \text{pathfinish } h$

6.10.3 Accessibility of frontier points

end

6.11 The Urysohn lemma, its consequences and other advanced material about metric spaces

theory *Urysohn*

imports *Abstract_Topological_Spaces Abstract_Metric_Spaces Infinite_Sum Arcwise_Connected*

begin

6.11.1 Urysohn lemma and Tietze's theorem

proposition *Urysohn_lemma*:

fixes $a \ b :: \text{real}$
assumes $\text{normal_space } X \text{ closedin } X \ S \text{ closedin } X \ T \text{ disjoint } S \ T \ a \leq b$
obtains f **where** $\text{continuous_map } X \ (\text{top_of_set } \{a..b\}) \ f \ f' \ S \subseteq \{a\} \ f' \ T \subseteq \{b\}$

theorem *Tietze_extension_closed_real_interval*:

assumes $\text{normal_space } X$ **and** $\text{closedin } X \ S$
and $\text{contf: continuous_map } (\text{subtopology } X \ S) \ \text{euclideanreal } f$
and $\text{fim: } f' \ S \subseteq \{a..b\}$ **and** $a \leq b$
obtains g
where $\text{continuous_map } X \ \text{euclideanreal } g$
 $\bigwedge x. x \in S \implies g \ x = f \ x \ g' \ \text{topspace } X \subseteq \{a..b\}$

theorem *Tietze_extension_realinterval*:

assumes $X \ S: \text{normal_space } X \text{ closedin } X \ S$ **and** $T: \text{is_interval } T \ T \neq \{\}$
and $\text{contf: continuous_map } (\text{subtopology } X \ S) \ \text{euclideanreal } f$
and $f' \ S \subseteq T$
obtains g **where** $\text{continuous_map } X \ \text{euclideanreal } g \ g' \ \text{topspace } X \subseteq T \ \bigwedge x. x \in S \implies g \ x = f \ x$

6.11.2 Random metric space stuff

6.11.3 Hereditarily normal spaces

6.11.4 Completely regular spaces

proposition *locally_compact_regular_imp_completely_regular_space:*

assumes *locally_compact_space X Hausdorff_space X \vee regular_space X*

shows *completely_regular_space X*

proposition *completely_regular_space_product_topology:*

completely_regular_space (product_topology X I) \longleftrightarrow

($\exists i \in I. X\ i = \text{trivial_topology}$) \vee ($\forall i \in I. \text{completely_regular_space } (X\ i)$)

(is ?lhs \longleftrightarrow ?rhs)

6.11.5 More generally, the k-ification functor

6.11.6 One-point compactifications and the Alexandroff extension construction

proposition *kc_space_one_point_compactification_gen:*

assumes *compact_space X*

shows *kc_space X \longleftrightarrow*

openin X (topspace X - {a}) \wedge ($\forall K. \text{compactin } X\ K \wedge a \notin K \longrightarrow \text{closedin } X\ K)$ \wedge

k_space (subtopology X (topspace X - {a})) \wedge kc_space (subtopology X (topspace X - {a}))

(is ?lhs \longleftrightarrow ?rhs)

proposition *istopology_Alexandroff_open: istopology (Alexandroff_open X)*

proposition *regular_space_one_point_compactification:*
assumes *compact_space X and ope: openin X (topspace X - {a})*
and \S : $\bigwedge K. \llbracket \text{compactin (subtopology X (topspace X - \{a\})) K}; \text{closedin (subtopology X (topspace X - \{a\})) K} \rrbracket \implies \text{closedin X K}$
shows *regular_space X \longleftrightarrow*
regular_space (subtopology X (topspace X - {a})) \wedge locally_compact_space (subtopology X (topspace X - {a}))
(is ?lhs \longleftrightarrow ?rhs)

proposition *Hausdorff_space_one_point_compactification_asymmetric_prod:*
assumes *compact_space X*
shows *Hausdorff_space X \longleftrightarrow*
kc_space (prod_topology X (subtopology X (topspace X - {a}))) \wedge
k_space (prod_topology X (subtopology X (topspace X - {a}))) (is ?lhs
 \longleftrightarrow ?rhs)

6.11.7 Extending continuous maps "pointwise" in a regular space

6.11.8 Extending Cauchy continuous functions to the closure

6.11.9 Metric space of bounded functions

- 6.11.10 Metric space of continuous bounded functions

- 6.11.11 Existence of completion for any metric space M as a subspace of $M \Rightarrow \mathbb{R}$

- 6.11.12 Contractions
- 6.11.13 The Baire Category Theorem

- 6.11.14 Sierpinski-Hausdorff type results about countable closed unions
- 6.11.15 The Tychonoff embedding

- 6.11.16 Urysohn and Tietze analogs for completely regular spaces

- 6.11.17 Size bounds on connected or path-connected spaces

- 6.11.18 Lavrentiev extension etc

6.11.19 Embedding in products and hence more about completely metrizable spaces

6.11.20 Theorems from Kuratowski

6.11.21 A perfect set in common cases must have at least the cardinality of the continuum

proposition *Kuratowski_component_number_invariance_aux:*

assumes *compact_space X and HsX: Hausdorff_space X*
and *lcX: locally_connected_space X and hnX: hereditarily_normal_space X*
and *hom: (subtopology X S) homeomorphic_space (subtopology X T)*
and *leXS: $\{.. $n::nat$ \} \lesssim \text{connected_components_of (subtopology X (topspace X - S))}$*
assumes $\S: \bigwedge S T.$
 $\llbracket \text{closedin } X S; \text{closedin } X T; (\text{subtopology } X S) \text{ homeomorphic_space (subtopology } X T);$
 $\{.. $n::nat$ \} \lesssim \text{connected_components_of (subtopology } X (\text{topspace } X - S)) \rrbracket$
 $\implies \{.. $n::nat$ \} \lesssim \text{connected_components_of (subtopology } X (\text{topspace } X - T))$
shows $\{.. $n::nat$ \} \lesssim \text{connected_components_of (subtopology } X (\text{topspace } X - T))$

theorem *Kuratowski_component_number_invariance:*

assumes *compact_space X Hausdorff_space X locally_connected_space X hereditarily_normal_space X*
shows $(\forall S T n.$
 $\text{closedin } X S \wedge \text{closedin } X T \wedge$
 $(\text{subtopology } X S) \text{ homeomorphic_space (subtopology } X T)$
 $\longrightarrow (\text{connected_components_of (subtopology } X (\text{topspace } X - S)) \approx \{.. $n::nat$ \} \longleftrightarrow$
 $\text{connected_components_of (subtopology } X (\text{topspace } X - T)) \approx \{.. $n::nat$ \}) \longleftrightarrow$
 $(\forall S T n.$
 $(\text{subtopology } X S) \text{ homeomorphic_space (subtopology } X T)$

100

```
      → (connected_components_of
         (subtopology X (topspace X - S)) ≈ {..n::nat} ↔
         connected_components_of
         (subtopology X (topspace X - T)) ≈ {..n::nat}))
    (is ?lhs = ?rhs)
```

```
end
theory Isolated
  imports HOL-Analysis.Elementary_Metric_Spaces
```

```
begin
```

6.11.22 Isolate and discrete

```
end
theory Sparse_In
  imports Homotopy
```

```
begin
```

6.11.23 A set of points sparse in another set

6.11.24 Co-sparseness filter

```
end
```

6.12 Operator Norm

```
theory Operator_Norm
  imports Complex_Main
begin
```

```
definition
```

```
onorm :: ('a::real_normed_vector ⇒ 'b::real_normed_vector) ⇒ real where
onorm f = (SUP x. norm (f x) / norm x)
```

```
proposition onorm_bound:
```

```
  assumes  $0 \leq b$  and  $\bigwedge x. \text{norm } (f x) \leq b * \text{norm } x$ 
  shows  $\text{onorm } f \leq b$ 
```

```
end
```

6.13 Limits on the Extended Real Number Line

```

theory Extended_Real_Limits
imports
  Topology_Euclidean_Space
  HOL-Library.Extended_Real
  HOL-Library.Extended_Nonnegative_Real
  HOL-Library.Indicator_Function
begin

```

6.13.1 Extended-Real.thy

Continuity of addition

Continuity of multiplication

Continuity of division

6.13.2 Extended-Nonnegative-Real.thy

6.13.3 monoset

6.13.4 Relate extended reals and the indicator function

end

6.14 Radius of Convergence and Summation Tests

```

theory Summation_Tests
imports
  Complex_Main
  HOL-Library.Discrete
  HOL-Library.Extended_Real
  HOL-Library.Liminf_Limsup
  Extended_Real_Limits
begin

```

6.14.1 Convergence tests for infinite sums

```

theorem root_test_convergence':
  fixes  $f :: \text{nat} \Rightarrow 'a :: \text{banach}$ 
  defines  $l \equiv \text{limsup } (\lambda n. \text{ereal } (\text{root } n \ (\text{norm } (f \ n))))$ 

```

assumes $l: l < 1$
shows *summable* f

theorem *root_test_divergence*:
fixes $f :: \text{nat} \Rightarrow 'a :: \text{banach}$
defines $l \equiv \text{limsup } (\lambda n. \text{ereal } (\text{root } n \text{ (norm } (f \ n))))$
assumes $l: l > 1$
shows $\neg \text{summable } f$

theorem *condensation_test*:
assumes *mono*: $\bigwedge m. 0 < m \implies f \text{ (Suc } m) \leq f \ m$
assumes *nonneg*: $\bigwedge n. f \ n \geq 0$
shows *summable* $f \longleftrightarrow \text{summable } (\lambda n. 2^{\wedge} n * f \ (2^{\wedge} n))$

theorem *summable_complex_powr_iff*:
assumes $\text{Re } s < -1$
shows *summable* $(\lambda n. \text{exp } (\text{of_real } (\ln \text{ (of_nat } n)) * s))$

theorem *kummers_test_convergence*:
fixes $f \ p :: \text{nat} \Rightarrow \text{real}$
assumes *pos_f*: *eventually* $(\lambda n. f \ n > 0)$ *sequentially*
assumes *nonneg_p*: *eventually* $(\lambda n. p \ n \geq 0)$ *sequentially*
defines $l \equiv \text{liminf } (\lambda n. \text{ereal } (p \ n * f \ n / f \text{ (Suc } n) - p \text{ (Suc } n)))$
assumes $l: l > 0$
shows *summable* f

theorem *kummers_test_divergence*:
fixes $f \ p :: \text{nat} \Rightarrow \text{real}$
assumes *pos_f*: *eventually* $(\lambda n. f \ n > 0)$ *sequentially*
assumes *pos_p*: *eventually* $(\lambda n. p \ n > 0)$ *sequentially*
assumes *divergent_p*: $\neg \text{summable } (\lambda n. \text{inverse } (p \ n))$
defines $l \equiv \text{limsup } (\lambda n. \text{ereal } (p \ n * f \ n / f \text{ (Suc } n) - p \text{ (Suc } n)))$
assumes $l: l < 0$
shows $\neg \text{summable } f$

theorem *ratio_test_convergence*:
fixes $f :: \text{nat} \Rightarrow \text{real}$
assumes *pos_f*: *eventually* $(\lambda n. f \ n > 0)$ *sequentially*
defines $l \equiv \text{liminf } (\lambda n. \text{ereal } (f \ n / f \text{ (Suc } n)))$
assumes $l: l > 1$
shows *summable* f

theorem *ratio_test_divergence*:
fixes $f :: \text{nat} \Rightarrow \text{real}$
assumes *pos_f*: *eventually* $(\lambda n. f \ n > 0)$ *sequentially*
defines $l \equiv \text{limsup } (\lambda n. \text{ereal } (f \ n / f \text{ (Suc } n)))$
assumes $l: l < 1$
shows $\neg \text{summable } f$

theorem *raabes_test_convergence*:
fixes $f :: \text{nat} \Rightarrow \text{real}$

```

assumes pos: eventually ( $\lambda n. f\ n > 0$ ) sequentially
defines l  $\equiv$  liminf ( $\lambda n. ereal\ (of\_nat\ n * (f\ n / f\ (Suc\ n) - 1))$ )
assumes l:  $l > 1$ 
shows summable f

```

theorem raabes_test_divergence:

```

fixes f :: nat  $\Rightarrow$  real
assumes pos: eventually ( $\lambda n. f\ n > 0$ ) sequentially
defines l  $\equiv$  limsup ( $\lambda n. ereal\ (of\_nat\ n * (f\ n / f\ (Suc\ n) - 1))$ )
assumes l:  $l < 1$ 
shows  $\neg$ summable f

```

6.14.2 Radius of convergence

definition conv_radius :: (nat \Rightarrow 'a :: banach) \Rightarrow ereal **where**
 conv_radius f = inverse (limsup ($\lambda n. ereal\ (root\ n\ (norm\ (f\ n)))$)))

theorem abs_summable_in_conv_radius:

```

fixes f :: nat  $\Rightarrow$  'a :: {banach, real_normed_div_algebra}
assumes ereal (norm z) < conv_radius f
shows summable ( $\lambda n. norm\ (f\ n * z^{\wedge} n)$ )

```

theorem not_summable_outside_conv_radius:

```

fixes f :: nat  $\Rightarrow$  'a :: {banach, real_normed_div_algebra}
assumes ereal (norm z) > conv_radius f
shows  $\neg$ summable ( $\lambda n. f\ n * z^{\wedge} n$ )

```

end

6.15 Uniform Limit and Uniform Convergence

theory Uniform_Limit

imports Connected_Summation_Tests Infinite_Sum

begin

6.15.1 Definition

definition uniformly_on :: 'a set \Rightarrow ('a \Rightarrow 'b::metric_space) \Rightarrow ('a \Rightarrow 'b) filter
where uniformly_on S l = (INF e \in {0 <..}. principal {f. $\forall x\in S. dist\ (f\ x)\ (l\ x) < e$ })

abbreviation

```

uniform_limit S f l  $\equiv$  filterlim f (uniformly_on S l)

```

proposition uniform_limit_iff:

```

uniform_limit S f l F  $\longleftrightarrow$  ( $\forall e > 0. \forall_F n\ in\ F. \forall x\in S. dist\ (f\ n\ x)\ (l\ x) < e$ )

```

6.15.2 Exchange limits

proposition *swap_uniform_limit*:
assumes $f: \forall_F n \text{ in } F. (f\ n \longrightarrow g\ n)$ (at x within S)
assumes $g: (g \longrightarrow l)$ F
assumes $uc: \text{uniform_limit } S\ f\ h\ F$
assumes $\neg \text{trivial_limit } F$
shows $(h \longrightarrow l)$ (at x within S)

6.15.3 Uniform limit theorem

theorem *uniform_limit_theorem*:
assumes $c: \forall_F n \text{ in } F. \text{continuous_on } A\ (f\ n)$
assumes $ul: \text{uniform_limit } A\ f\ l\ F$
assumes $\neg \text{trivial_limit } F$
shows $\text{continuous_on } A\ l$

6.15.4 Comparison Test

6.15.5 Weierstrass M-Test

proposition *Weierstrass_m_test_ev*:
fixes $f :: _ \Rightarrow _ \Rightarrow _ :: \text{banach}$
assumes $\text{eventually } (\lambda n. \forall x \in A. \text{norm } (f\ n\ x) \leq M\ n)$ *sequentially*
assumes $\text{summable } M$
shows $\text{uniform_limit } A\ (\lambda n\ x. \sum_{i < n}. f\ i\ x)$ $(\lambda x. \text{suminf } (\lambda i. f\ i\ x))$ *sequentially*

6.15.6 Power series and uniform convergence

proposition *power_uniformly_convergent*:
fixes $a :: \text{nat} \Rightarrow 'a :: \{\text{real_normed_div_algebra}, \text{banach}\}$
assumes $r < \text{conv_radius } a$
shows $\text{uniformly_convergent_on } (\text{cball } \xi\ r)$ $(\lambda n\ x. \sum_{i < n}. a\ i * (x - \xi) ^ i)$

end

6.16 Bounded Linear Function

theory *Bounded_Linear_Function*
imports
Topology_Euclidean_Space
Operator_Norm
Uniform_Limit

Function_Topology

begin

6.16.1 Type of bounded linear functions

```
typedef (overloaded) ('a, 'b) blinfun (( $\_ \Rightarrow_L \_$ ) [22, 21] 21) =
  {f::'a::real_normed_vector $\Rightarrow$ 'b::real_normed_vector. bounded_linear f}
morphisms blinfun_apply Blinfun
```

6.16.2 Type class instantiations

```
instantiation blinfun :: (real_normed_vector, real_normed_vector) real_normed_vector
begin
```

```
lift_definition norm_blinfun :: 'a  $\Rightarrow_L$  'b  $\Rightarrow$  real is onorm
```

```
lift_definition zero_blinfun :: 'a  $\Rightarrow_L$  'b is  $\lambda x. 0$ 
```

```
lift_definition plus_blinfun :: 'a  $\Rightarrow_L$  'b  $\Rightarrow$  'a  $\Rightarrow_L$  'b  $\Rightarrow$  'a  $\Rightarrow_L$  'b
is  $\lambda f g x. f x + g x$ 
```

```
lift_definition scaleR_blinfun::real  $\Rightarrow$  'a  $\Rightarrow_L$  'b  $\Rightarrow$  'a  $\Rightarrow_L$  'b is  $\lambda r f x. r *_R f x$ 
```

6.16.3 The strong operator topology on continuous linear operators

```
definition strong_operator_topology::('a::real_normed_vector  $\Rightarrow_L$  'b::real_normed_vector)
topology
```

```
where strong_operator_topology = pullback_topology UNIV blinfun_apply euclidean
```

end

6.17 Derivative

```
theory Derivative
```

```
imports
```

```
  Bounded_Linear_Function
```

```
  Line_Segment
```

Convex_Euclidean_Space
begin

6.17.1 Derivatives

proposition *has_derivative_within'*:

$$(f \text{ has_derivative } f')(at\ x\ \text{within } s) \longleftrightarrow$$

$$\text{bounded_linear } f' \wedge$$

$$(\forall e > 0. \exists d > 0. \forall x' \in s. 0 < \text{norm } (x' - x) \wedge \text{norm } (x' - x) < d \longrightarrow$$

$$\text{norm } (f\ x' - f\ x - f'(x' - x)) / \text{norm } (x' - x) < e)$$

6.17.2 Differentiability

definition

differentiable_on :: ('a::real_normed_vector \Rightarrow 'b::real_normed_vector) \Rightarrow 'a set
 \Rightarrow bool

(**infix** *differentiable'_on* 50)

where *f differentiable_on* s \longleftrightarrow ($\forall x \in s. f$ differentiable (at x within s))

6.17.3 Frechet derivative and Jacobian matrix

proposition *frechet_derivative_works*:

f differentiable net \longleftrightarrow (*f has_derivative* (*frechet_derivative* f net)) net

6.17.4 Differentiability implies continuity

proposition *differentiable_imp_continuous_within*:

f differentiable (at x within s) \implies *continuous* (at x within s) f

6.17.5 The chain rule

proposition *diff_chain_within*[*derivative_intros*]:

assumes (*f has_derivative* f') (at x within s)

and (*g has_derivative* g') (at (f x) within (f ' s))

shows ((g \circ f) *has_derivative* (g' \circ f'))(at x within s)

6.17.6 Uniqueness of derivative

The general result is a bit messy because we need approachability of the limit point from any direction. But OK for nontrivial intervals etc.

proposition *frechet_derivative_unique_within*:

fixes f :: 'a::euclidean_space \Rightarrow 'b::real_normed_vector

assumes 1: $(f \text{ has_derivative } f')$ (at x within S)
and 2: $(f \text{ has_derivative } f'')$ (at x within S)
and $S: \bigwedge i e. \llbracket i \in \text{Basis}; e > 0 \rrbracket \implies \exists d. 0 < |d| \wedge |d| < e \wedge (x + d *_{\mathbb{R}} i) \in S$
shows $f' = f''$

proposition *frechet_derivative_unique_within_closed_interval*:

fixes $f :: 'a :: \text{euclidean_space} \Rightarrow 'b :: \text{real_normed_vector}$
assumes $ab: \bigwedge i. i \in \text{Basis} \implies a \cdot i < b \cdot i$
and $x: x \in \text{cbox } a \ b$
and $(f \text{ has_derivative } f')$ (at x within $\text{cbox } a \ b$)
and $(f \text{ has_derivative } f'')$ (at x within $\text{cbox } a \ b$)
shows $f' = f''$

6.17.7 Derivatives of local minima and maxima are zero

6.17.8 One-dimensional mean value theorem

6.17.9 More general bound theorems

proposition *differentiable_bound_general*:

fixes $f :: \text{real} \Rightarrow 'a :: \text{real_normed_vector}$
assumes $a < b$
and $f_cont: \text{continuous_on } \{a..b\} \ f$
and $phi_cont: \text{continuous_on } \{a..b\} \ \varphi$
and $f': \bigwedge x. a < x \implies x < b \implies (f \text{ has_vector_derivative } f' \ x)$ (at x)
and $phi': \bigwedge x. a < x \implies x < b \implies (\varphi \text{ has_vector_derivative } \varphi' \ x)$ (at x)
and $bnd: \bigwedge x. a < x \implies x < b \implies \text{norm } (f' \ x) \leq \varphi' \ x$
shows $\text{norm } (f \ b - f \ a) \leq \varphi \ b - \varphi \ a$

6.17.10 Differentiability of inverse function (most basic form)

proposition *has_derivative_inverse*:

fixes $f :: 'a :: \text{real_normed_vector} \Rightarrow 'b :: \text{real_normed_vector}$
assumes *compact* S
and $x \in S$
and $fx: f \ x \in \text{interior } (f^{-1} \ S)$
and $\text{continuous_on } S \ f$
and $gf: \bigwedge y. y \in S \implies g \ (f \ y) = y$
and $B: (f \text{ has_derivative } f')$ (at x) $\text{bounded_linear } g' \ g' \circ f' = \text{id}$
shows $(g \text{ has_derivative } g')$ (at $(f \ x)$)

proposition *has_derivative_locally_injective*:

fixes $f :: 'n :: \text{euclidean_space} \Rightarrow 'm :: \text{euclidean_space}$
assumes $a \in S$
and *open* S
and $blig: \text{bounded_linear } g'$
and $g' \circ f' \ a = \text{id}$
and $\text{derf}: \bigwedge x. x \in S \implies (f \text{ has_derivative } f' \ x)$ (at x)
and $\bigwedge e. e > 0 \implies \exists d > 0. \forall x. \text{dist } a \ x < d \implies \text{onorm } (\lambda v. f' \ x \ v - f' \ a \ v)$

$< e$
obtains r where $r > 0$ ball a $r \subseteq S$ inj_on f (ball a r)

6.17.11 Uniformly convergent sequence of derivatives

proposition *has_derivative_sequence*:

fixes $f :: \text{nat} \Rightarrow 'a::\text{real_normed_vector} \Rightarrow 'b::\text{banach}$

assumes *convex* S

and *derf*: $\bigwedge n x. x \in S \implies ((f\ n) \text{ has_derivative } (f'\ n\ x))$ (at x within S)

and *nle*: $\bigwedge e. e > 0 \implies \forall_F n$ in sequentially. $\forall x \in S. \forall h. \text{norm } (f'\ n\ x\ h - g'$
 $x\ h) \leq e * \text{norm } h$

and $x0 \in S$

and *lim*: $(\lambda n. f\ n\ x0) \longrightarrow l$ sequentially

shows $\exists g. \forall x \in S. (\lambda n. f\ n\ x) \longrightarrow g\ x \wedge (g \text{ has_derivative } g'(x))$ (at x within S)

6.17.12 Differentiation of a series

proposition *has_derivative_series*:

fixes $f :: \text{nat} \Rightarrow 'a::\text{real_normed_vector} \Rightarrow 'b::\text{banach}$

assumes *convex* S

and $\bigwedge n x. x \in S \implies ((f\ n) \text{ has_derivative } (f'\ n\ x))$ (at x within S)

and $\bigwedge e. e > 0 \implies \forall_F n$ in sequentially. $\forall x \in S. \forall h. \text{norm } (\text{sum } (\lambda i. f'\ i\ x\ h)$
 $\{..<n\} - g'\ x\ h) \leq e * \text{norm } h$

and $x \in S$

and $(\lambda n. f\ n\ x)$ sums l

shows $\exists g. \forall x \in S. (\lambda n. f\ n\ x) \text{ sums } (g\ x) \wedge (g \text{ has_derivative } g'\ x)$ (at x within S)

6.17.13 Derivative as a vector

proposition *vector_derivative_works*:

f differentiable net $\longleftrightarrow (f \text{ has_vector_derivative } (\text{vector_derivative } f\ \text{net}))$ net
 (is $?l = ?r$)

6.17.14 Field differentiability

definition *field_differentiable* :: $['a \Rightarrow 'a::\text{real_normed_field}, 'a\ \text{filter}] \Rightarrow \text{bool}$

(**infixr** (*field'_differentiable*) 50)

where $f \text{ field_differentiable } F \equiv \exists f'. (f \text{ has_field_derivative } f')\ F$

6.17.15 Field derivative

definition $deriv :: ('a \Rightarrow 'a::real_normed_field) \Rightarrow 'a \Rightarrow 'a$ **where**
 $deriv\ f\ x \equiv SOME\ D.\ DERIV\ f\ x\ :>\ D$

proposition $field_differentiable_derivI$:

$f\ field_differentiable\ (at\ x) \implies (f\ has_field_derivative\ deriv\ f\ x)\ (at\ x)$

6.17.16 Relation between convexity and derivative

proposition $convex_on_imp_above_tangent$:

assumes $convex$: $convex_on\ A\ f$ **and** $connected$: $connected\ A$

assumes c : $c \in interior\ A$ **and** x : $x \in A$

assumes $deriv$: $(f\ has_field_derivative\ f')$ $(at\ c\ within\ A)$

shows $f\ x - f\ c \geq f' * (x - c)$

6.17.17 Partial derivatives

proposition $has_derivative_partialsI$:

fixes $f :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector \Rightarrow 'c::real_normed_vector$

assumes fx : $((\lambda x.\ f\ x\ y)\ has_derivative\ fx)\ (at\ x\ within\ X)$

assumes fy : $\bigwedge x\ y.\ x \in X \implies y \in Y \implies ((\lambda y.\ f\ x\ y)\ has_derivative\ blinfun_apply\ (fy\ x\ y))\ (at\ y\ within\ Y)$

assumes fy_cont [$unfolded\ continuous_within$]: $continuous\ (at\ (x,\ y)\ within\ X \times Y)\ (\lambda(x,\ y).\ fy\ x\ y)$

assumes $y \in Y$ $convex\ Y$

shows $((\lambda(x,\ y).\ f\ x\ y)\ has_derivative\ (\lambda(tx,\ ty).\ fx\ tx + fy\ x\ y\ ty))\ (at\ (x,\ y)\ within\ X \times Y)$

6.17.18 The Inverse Function Theorem

theorem $inverse_function_theorem$:

fixes $f :: 'a::euclidean_space \Rightarrow 'a$

and $f' :: 'a \Rightarrow ('a \Rightarrow_L 'a)$

assumes $open\ U$

and $deriv$: $\bigwedge x.\ x \in U \implies (f\ has_derivative\ (blinfun_apply\ (f'\ x)))\ (at\ x)$

and $contf$: $continuous_on\ U\ f'$

and $x0 \in U$

and $invf$: $invf\ o_L\ f'\ x0 = id_blinfun$

obtains $U' V g g'$ **where** $open\ U'\ U' \subseteq U\ x0 \in U'\ open\ V\ f\ x0 \in V$ *homeomorphism* $U' V f g$

$\bigwedge y.\ y \in V \implies (g\ has_derivative\ (g'\ y))\ (at\ y)$

$\bigwedge y.\ y \in V \implies g'\ y = inv\ (blinfun_apply\ (f'(g\ y)))$

$\bigwedge y.\ y \in V \implies bij\ (blinfun_apply\ (f'(g\ y)))$

6.17.19 The concept of continuously differentiable

definition *C1_differentiable_on* :: (real \Rightarrow 'a::real_normed_vector) \Rightarrow real set \Rightarrow bool

(**infix** *C1'_differentiable'_on* 50)

where

f C1_differentiable_on S \longleftrightarrow

($\exists D. (\forall x \in S. (f \text{ has_vector_derivative } (D x)) \text{ (at } x)) \wedge \text{continuous_on } S D$)

definition *piecewise_C1_differentiable_on*

(**infix** *piecewise'_C1'_differentiable'_on* 50)

where *f piecewise_C1_differentiable_on i* \equiv

continuous_on i f \wedge

($\exists S. \text{finite } S \wedge (f \text{ C1_differentiable_on } (i - S))$)

end

6.18 Finite Cartesian Products of Euclidean Spaces

theory *Cartesian_Euclidean_Space*

imports *Derivative*

begin

6.18.1 Closures and interiors of halfspaces

6.18.2 Bounds on components etc. relative to operator norm

6.18.3 Convex Euclidean Space

6.18.4 Arbitrarily good rational approximations

proposition *matrix_rational_approximation*:

fixes *A* :: realⁿ^m

assumes *e* > 0

obtains *B* where $\bigwedge i j. B_{ij} \in \mathbb{Q} \text{ onorm}(\lambda x. (A - B) * v x) < e$

6.18.5 Derivative

definition *jacobian f net* = *matrix(frechet_derivative f net)*

proposition *jacobian_works*:

(*f*::(real^a) \Rightarrow (real^b)) *differentiable net* \longleftrightarrow

(*f* has_derivative ($\lambda h. (\text{jacobian } f \text{ net}) * v h$)) net (is ?lhs = ?rhs)
proposition *differential_zero_maxmin_cart*:
 fixes $f :: \text{real}^a \Rightarrow \text{real}^b$
 assumes $0 < e ((\forall y \in \text{ball } x \ e. (f y)\$k \leq (f x)\$k) \vee (\forall y \in \text{ball } x \ e. (f x)\$k \leq (f y)\$k))$
f differentiable (at x)
 shows *jacobian f (at x) \$ k = 0*

end

6.19 Complex Analysis Basics

theory *Complex_Analysis_Basics*
 imports *Derivative HOL-Library.Nonpos_Ints Uncountable_Sets*
 begin

6.19.1 Holomorphic functions

definition *holomorphic_on* :: [*complex* \Rightarrow *complex*, *complex set*] \Rightarrow *bool*
 (infixl (*holomorphic'_on*) 50)
 where *f holomorphic_on s* $\equiv \forall x \in s. f \text{ field_differentiable (at } x \text{ within } s)$

named_theorems *holomorphic_intros* *structural introduction rules for holomorphic_on*

6.19.2 Analyticity on a set

definition *analytic_on* (infixl (*analytic'_on*) 50)
 where *f analytic_on S* $\equiv \forall x \in S. \exists \varepsilon. 0 < \varepsilon \wedge f \text{ holomorphic_on (ball } x \ \varepsilon)$

named_theorems *analytic_intros* *introduction rules for proving analyticity*

end

6.20 Complex Transcendental Functions

theory *Complex_Transcendental*
 imports
Complex_Analysis_Basics Summation_Tests HOL-Library.Periodic_Fun
 begin

6.20.1 Möbius transformations

definition *moebius a b c d* $\equiv (\lambda z. (a*z+b) / (c*z+d :: 'a :: \text{field}))$

theorem *moebius_inverse*:
 assumes $a * d \neq b * c$ $c * z + d \neq 0$

shows $moebius\ d\ (-b)\ (-c)\ a\ (moebius\ a\ b\ c\ d\ z) = z$

6.20.2 Euler and de Moivre formulas

theorem *exp_Euler*: $exp(i * z) = cos(z) + i * sin(z)$

theorem *Euler*: $exp(z) = of_real(exp(Re\ z)) * (of_real(cos(Im\ z)) + i * of_real(sin(Im\ z)))$

6.20.3 The argument of a complex number (HOL Light version)

definition *is_Arg* :: $[complex, real] \Rightarrow bool$
 where $is_Arg\ z\ r \equiv z = of_real(norm\ z) * exp(i * of_real\ r)$

definition *Arg2pi* :: $complex \Rightarrow real$
 where $Arg2pi\ z \equiv if\ z = 0\ then\ 0\ else\ THE\ t.\ 0 \leq t \wedge t < 2 * pi \wedge is_Arg\ z\ t$

6.20.4 The principal branch of the Complex logarithm

instantiation *complex* :: *ln*
begin

definition *ln_complex* :: $complex \Rightarrow complex$
 where $ln_complex \equiv \lambda z.\ THE\ w.\ exp\ w = z \ \&\ -pi < Im(w) \ \&\ Im(w) \leq pi$

theorem *Ln_series*:
fixes $z :: complex$
assumes $norm\ z < 1$
shows $(\lambda n.\ (-1)^{Suc\ n} / of_nat\ n * z^{n})\ sums\ ln\ (1 + z)$ (**is** $(\lambda n.\ ?f\ n * z^{n})\ sums\ _)$)

corollary *norm_Ln_prod_le*:
fixes $f :: 'a \Rightarrow complex$
assumes $\bigwedge x. x \in A \implies f\ x \neq 0$
shows $cmod\ (Ln\ (prod\ f\ A)) \leq (\sum\ x \in A.\ cmod\ (Ln\ (f\ x)))$

6.20.5 The Argument of a Complex Number

lemma *Arg_def*:
shows $Arg\ z = (if\ z = 0\ then\ 0\ else\ Im\ (Ln\ z))$

6.20.6 The Unwinding Number and the Ln product Formula

definition *unwinding* :: *complex* \Rightarrow *int* **where**

unwinding $z \equiv$ THE k . of_int $k = (z - \text{Ln}(\exp z)) / (\text{of_real}(2 * \pi) * i)$

6.20.7 Characterisation of $\text{Im}(\text{Ln } z)$ (Wenda Li)

6.20.8 Complex arctangent

definition *Arctan* :: *complex* \Rightarrow *complex* **where**

Arctan $\equiv \lambda z. (i/2) * \text{Ln}((1 - i*z) / (1 + i*z))$

theorem *Arctan_series*:

assumes $z: \text{norm } (z :: \text{complex}) < 1$

defines $g \equiv \lambda n. \text{if odd } n \text{ then } -i * i^n / n \text{ else } 0$

defines $h \equiv \lambda z n. (-1)^n / \text{of_nat } (2*n+1) * (z :: \text{complex})^{(2*n+1)}$

shows $(\lambda n. g n * z^n)$ sums *Arctan* z

and $h z$ sums *Arctan* z

theorem *ln_series_quadratic*:

assumes $x: x > (0 :: \text{real})$

shows $(\lambda n. (2*((x - 1) / (x + 1))^{(2*n+1)} / \text{of_nat } (2*n+1)))$ sums $\ln x$

6.20.9 Inverse Sine

definition *Arcsin* :: *complex* \Rightarrow *complex* **where**

Arcsin $\equiv \lambda z. -i * \text{Ln}(i * z + \text{csqrt}(1 - z^2))$

6.20.10 Inverse Cosine

definition *Arccos* :: *complex* \Rightarrow *complex* **where**

Arccos $\equiv \lambda z. -i * \text{Ln}(z + i * \text{csqrt}(1 - z^2))$

6.20.11 Roots of unity

theorem *complex_root_unity*:

fixes $j :: \text{nat}$

assumes $n \neq 0$

shows $\exp(2 * \text{of_real } \pi * i * \text{of_nat } j / \text{of_nat } n)^n = 1$

corollary *bij_betw_roots_unity*:

bij_betw $(\lambda j. \exp(2 * \text{of_real } \pi * i * \text{of_nat } j / \text{of_nat } n))$

$\{..<n\} \{ \exp(2 * \text{of_real } \pi * i * \text{of_nat } j / \text{of_nat } n) \mid j. j < n \}$

end

Chapter 7

Measure and Integration Theory

```
theory Sigma_Algebra
imports
  Complex_Main
  HOL-Library.Countable_Set
  HOL-Library.FuncSet
  HOL-Library.Indicator_Function
  HOL-Library.Extended_Nonnegative_Real
  HOL-Library.Disjoint_Sets
begin
```

7.1 Sigma Algebra

7.1.1 Families of sets

```
locale subset_class =
  fixes  $\Omega :: 'a \text{ set}$  and  $M :: 'a \text{ set set}$ 
  assumes space_closed:  $M \subseteq \text{Pow } \Omega$ 
locale semiring_of_sets = subset_class +
  assumes empty_sets_iff:  $\{\} \in M$ 
  assumes Int[intro]:  $\bigwedge a b. a \in M \implies b \in M \implies a \cap b \in M$ 
  assumes Diff_cover:
     $\bigwedge a b. a \in M \implies b \in M \implies \exists C \subseteq M. \text{finite } C \wedge \text{disjoint } C \wedge a - b = \bigcup C$ 
locale ring_of_sets = semiring_of_sets +
  assumes Un [intro]:  $\bigwedge a b. a \in M \implies b \in M \implies a \cup b \in M$ 
locale algebra = ring_of_sets +
  assumes top [iff]:  $\Omega \in M$ 
```

```
proposition algebra_iff_Un:
  algebra  $\Omega$   $M \iff$ 
     $M \subseteq \text{Pow } \Omega \wedge$ 
     $\{\} \in M \wedge$ 
     $(\forall a \in M. \Omega - a \in M) \wedge$ 
```

$$(\forall a \in M. \forall b \in M. a \cup b \in M) \text{ (is_} _ \longleftrightarrow ?Un)$$

proposition *algebra_iff_Int*:

$$\begin{aligned} & algebra \ \Omega \ M \longleftrightarrow \\ & M \subseteq Pow \ \Omega \ \& \ \{\} \in M \ \& \\ & (\forall a \in M. \ \Omega - a \in M) \ \& \\ & (\forall a \in M. \ \forall b \in M. \ a \cap b \in M) \text{ (is_} _ \longleftrightarrow ?Int) \end{aligned}$$

locale *sigma_algebra* = *algebra* +

$$\text{assumes } countable_nat_UN \ [intro]: \bigwedge A. \ range \ A \subseteq M \implies (\bigcup i::nat. \ A \ i) \in M$$

Sigma algebras can naturally be created as the closure of any set of M with regard to the properties just postulated.

inductive_set *sigma_sets* :: 'a set \Rightarrow 'a set set \Rightarrow 'a set set

for *sp* :: 'a set **and** *A* :: 'a set set

where

$$\begin{aligned} & Basic[intro, simp]: a \in A \implies a \in sigma_sets \ sp \ A \\ & | Empty: \{\} \in sigma_sets \ sp \ A \\ & | Compl: a \in sigma_sets \ sp \ A \implies sp - a \in sigma_sets \ sp \ A \\ & | Union: (\bigwedge i::nat. \ a \ i \in sigma_sets \ sp \ A) \implies (\bigcup i. \ a \ i) \in sigma_sets \ sp \ A \end{aligned}$$

definition *closed_cdi* :: 'a set \Rightarrow 'a set set \Rightarrow bool **where**

$$\begin{aligned} & closed_cdi \ \Omega \ M \longleftrightarrow \\ & M \subseteq Pow \ \Omega \ \& \\ & (\forall s \in M. \ \Omega - s \in M) \ \& \\ & (\forall A. \ (range \ A \subseteq M) \ \& \ (A \ 0 = \{\}) \ \& \ (\forall n. \ A \ n \subseteq A \ (Suc \ n)) \longrightarrow \\ & \quad (\bigcup i. \ A \ i) \in M) \ \& \\ & (\forall A. \ (range \ A \subseteq M) \ \& \ disjoint_family \ A \longrightarrow (\bigcup i::nat. \ A \ i) \in M) \end{aligned}$$

locale *Dynkin_system* = *subset_class* +

assumes *space*: $\Omega \in M$

and *compl*[intro!]: $\bigwedge A. \ A \in M \implies \Omega - A \in M$

and *UN*[intro!]: $\bigwedge A. \ disjoint_family \ A \implies range \ A \subseteq M \implies (\bigcup i::nat. \ A \ i) \in M$

definition *Int_stable* :: 'a set set \Rightarrow bool **where**

$$Int_stable \ M \longleftrightarrow (\forall a \in M. \ \forall b \in M. \ a \cap b \in M)$$

definition *Dynkin* :: 'a set \Rightarrow 'a set set \Rightarrow 'a set set **where**

$$Dynkin \ \Omega \ M = (\bigcap \{D. \ Dynkin_system \ \Omega \ D \wedge M \subseteq D\})$$

The reason to introduce Dynkin-systems is the following induction rules for σ -algebras generated by a generator closed under intersection.

proposition *sigma_sets_induct_disjoint*[consumes 3, case_names basic empty compl union]:

assumes *Int_stable* *G*

and *closed*: $G \subseteq Pow \ \Omega$

and *A*: $A \in sigma_sets \ \Omega \ G$

assumes *basic*: $\bigwedge A. \ A \in G \implies P \ A$

and *empty*: $P \ \{\}$

and *compl*: $\bigwedge A. \ A \in sigma_sets \ \Omega \ G \implies P \ A \implies P \ (\Omega - A)$

and union: $\bigwedge A. \text{disjoint_family } A \implies \text{range } A \subseteq \text{sigma_sets } \Omega \ G \implies (\bigwedge i. P (A\ i)) \implies P (\bigcup i::\text{nat}. A\ i)$
shows $P\ A$

7.1.2 Measure type

definition *positive* :: 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow bool **where**
positive $M\ \mu \longleftrightarrow \mu\ \{\} = 0$

definition *countably_additive* :: 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow bool **where**
countably_additive $M\ f \longleftrightarrow$
 $(\forall A. \text{range } A \subseteq M \longrightarrow \text{disjoint_family } A \longrightarrow (\bigcup i. A\ i) \in M \longrightarrow$
 $(\sum i. f (A\ i)) = f (\bigcup i. A\ i))$

definition *measure_space* :: 'a set \Rightarrow 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow bool **where**
measure_space $\Omega\ A\ \mu \longleftrightarrow$
 $\text{sigma_algebra } \Omega\ A \wedge \text{positive } A\ \mu \wedge \text{countably_additive } A\ \mu$

typedef 'a measure =
 $\{(\Omega::'a\ \text{set}, A, \mu). (\forall a \in -A. \mu\ a = 0) \wedge \text{measure_space } \Omega\ A\ \mu\}$

definition *space* :: 'a measure \Rightarrow 'a set **where**
space $M = \text{fst } (\text{Rep_measure } M)$

definition *sets* :: 'a measure \Rightarrow 'a set set **where**
sets $M = \text{fst } (\text{snd } (\text{Rep_measure } M))$

definition *emeasure* :: 'a measure \Rightarrow 'a set \Rightarrow ennreal **where**
emeasure $M = \text{snd } (\text{snd } (\text{Rep_measure } M))$

definition *measure* :: 'a measure \Rightarrow 'a set \Rightarrow real **where**
measure $M\ A = \text{enn2real } (\text{emeasure } M\ A)$

definition *measure_of* :: 'a set \Rightarrow 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow 'a measure **where**
measure_of $\Omega\ A\ \mu =$
 $\text{Abs_measure } (\Omega, \text{if } A \subseteq \text{Pow } \Omega \text{ then } \text{sigma_sets } \Omega\ A \text{ else } \{\{\}, \Omega\},$
 $\lambda a. \text{if } a \in \text{sigma_sets } \Omega\ A \wedge \text{measure_space } \Omega\ (\text{sigma_sets } \Omega\ A) \ \mu \text{ then } \mu\ a$
 $\text{else } 0)$

proposition *emeasure_measure_of*:

assumes $M: M = \text{measure_of } \Omega\ A\ \mu$

assumes $ms: A \subseteq \text{Pow } \Omega \ \text{positive } (\text{sets } M) \ \mu \ \text{countably_additive } (\text{sets } M) \ \mu$

assumes $X: X \in \text{sets } M$

shows $\text{emeasure } M\ X = \mu\ X$

definition *measurable* :: 'a measure \Rightarrow 'b measure \Rightarrow ('a \Rightarrow 'b) set
(infixr \rightarrow_M 60) **where**

measurable $A\ B = \{f \in \text{space } A \rightarrow \text{space } B. \forall y \in \text{sets } B. f\ -'y \cap \text{space } A \in \text{sets}$

$A\}$
definition *count_space* :: 'a set \Rightarrow 'a measure **where**
count_space $\Omega = \text{measure_of } \Omega \text{ (Pow } \Omega \text{) } (\lambda A. \text{ if finite } A \text{ then of_nat (card } A \text{) else } \infty)$

7.1.3 The smallest σ -algebra regarding a function

definition *vimage_algebra* :: 'a set \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b measure \Rightarrow 'a measure
where
vimage_algebra $X f M = \text{sigma } X \{f - ' A \cap X \mid A. A \in \text{sets } M\}$

end

7.2 Measurability Prover

theory *Measurable*
imports
 Sigma_Algebra
 HOL-Library.Order_Continuity
begin

method_setup *measurable* = $\langle \text{Scan.lift (Scan.succeed (METHOD o Measurable.measurable_tac))} \rangle$
 measurability prover

simproc_setup *measurable* ($A \in \text{sets } M \mid f \in \text{measurable } M N$) =
 $\langle K \text{ Measurable.proc} \rangle$

end

7.3 Measure Spaces

theory *Measure_Space*
imports
 Measurable HOL-Library.Extended_Nonnegative_Real
begin

7.3.1 μ -null sets

definition *null_sets* :: 'a measure \Rightarrow 'a set set **where**
null_sets $M = \{N \in \text{sets } M. \text{emeasure } M N = 0\}$

7.3.2 The almost everywhere filter (i.e. quantifier)

definition *ae_filter* :: 'a measure \Rightarrow 'a filter **where**
ae_filter $M = (\text{INF } N \in \text{null_sets } M. \text{principal (space } M - N))$

7.3.3 σ -finite Measures

locale *sigma_finite_measure* =
fixes $M :: 'a \text{ measure}$
assumes *sigma_finite_countable*:
 $\exists A :: 'a \text{ set set. countable } A \wedge A \subseteq \text{sets } M \wedge (\bigcup A) = \text{space } M \wedge (\forall a \in A. \text{emeasure } M a \neq \infty)$

7.3.4 Measure space induced by distribution of (\rightarrow_M) -functions

definition *distr* :: $'a \text{ measure} \Rightarrow 'b \text{ measure} \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b \text{ measure}$ **where**
distr $M N f =$
measure_of (*space* N) (*sets* N) ($\lambda A. \text{emeasure } M (f \text{ - ' } A \cap \text{space } M)$)

proposition *distr_distr*:

$g \in \text{measurable } N L \Longrightarrow f \in \text{measurable } M N \Longrightarrow \text{distr } (\text{distr } M N f) L g = \text{distr } M L (g \circ f)$

7.3.5 Set of measurable sets with finite measure

definition *fmeasurable* :: $'a \text{ measure} \Rightarrow 'a \text{ set set}$ **where**
fmeasurable $M = \{A \in \text{sets } M. \text{emeasure } M A < \infty\}$

7.3.6 Measure spaces with $\text{emeasure } M (\text{space } M) < \infty$

locale *finite_measure* = *sigma_finite_measure* M **for** $M +$
assumes *finite_emeasure_space*: $\text{emeasure } M (\text{space } M) \neq \text{top}$

7.3.7 Scaling a measure

definition *scale_measure* :: $\text{ennreal} \Rightarrow 'a \text{ measure} \Rightarrow 'a \text{ measure}$ **where**
scale_measure $r M = \text{measure_of } (\text{space } M) (\text{sets } M) (\lambda A. r * \text{emeasure } M A)$

7.3.8 Complete lattice structure on measures

proposition *unsigned_Hahn_decomposition*:

assumes [*simp*]: $\text{sets } N = \text{sets } M$ **and** [*measurable*]: $A \in \text{sets } M$
and [*simp*]: $\text{emeasure } M A \neq \text{top}$ $\text{emeasure } N A \neq \text{top}$
shows $\exists Y \in \text{sets } M. Y \subseteq A \wedge (\forall X \in \text{sets } M. X \subseteq Y \longrightarrow N X \leq M X) \wedge (\forall X \in \text{sets } M. X \subseteq A \longrightarrow X \cap Y = \{\} \longrightarrow M X \leq N X)$

Define a lexicographical order on *measure*, in the order space, sets and measure. The parts of the lexicographical order are point-wise ordered.

instantiation *measure* :: (type) *order_bot*
begin

definition *less_measure* :: 'a *measure* \Rightarrow 'a *measure* \Rightarrow bool **where**
less_measure M N \longleftrightarrow (M \leq N \wedge \neg N \leq M)

definition *bot_measure* :: 'a *measure* **where**
bot_measure = *sigma* {} {}

proposition *le_measure*: sets M = sets N \implies M \leq N \longleftrightarrow (\forall A \in sets M. *emeasure* M A \leq *emeasure* N A)

definition *sup_measure'* :: 'a *measure* \Rightarrow 'a *measure* \Rightarrow 'a *measure* **where**
sup_measure' A B =
measure_of (space A) (sets A)
(λ X. SUP Y \in sets A. *emeasure* A (X \cap Y) + *emeasure* B (X \cap - Y))

definition *sup_lexord* :: 'a \Rightarrow 'a \Rightarrow ('a \Rightarrow 'b::order) \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a **where**
sup_lexord A B k s c =
(if k A = k B then c else
if \neg k A \leq k B \wedge \neg k B \leq k A then s else
if k B \leq k A then A else B)

instantiation *measure* :: (type) *semilattice_sup*
begin

definition *sup_measure* :: 'a *measure* \Rightarrow 'a *measure* \Rightarrow 'a *measure* **where**
sup_measure A B =
sup_lexord A B space (*sigma* (space A \cup space B) {})
(*sup_lexord* A B sets (*sigma* (space A) (sets A \cup sets B)) (*sup_measure'* A B))

definition
Sup_lexord :: ('a \Rightarrow 'b::complete_lattice) \Rightarrow ('a set \Rightarrow 'a) \Rightarrow ('a set \Rightarrow 'a) \Rightarrow 'a set \Rightarrow 'a

where
Sup_lexord k c s A =
(let U = (SUP a \in A. k a)
in if \exists a \in A. k a = U then c {a \in A. k a = U} else s A)

instantiation *measure* :: (type) *complete_lattice*
begin

definition *Sup_measure'* :: 'a *measure* set \Rightarrow 'a *measure* **where**
Sup_measure' M =
measure_of (\bigcup a \in M. space a) (\bigcup a \in M. sets a)
(λ X. (SUP P \in {P. finite P \wedge P \subseteq M }. *sup_measure.F* id P X))

definition *Sup_measure* :: 'a measure set \Rightarrow 'a measure **where**

Sup_measure =
Sup_lexord space
 (*Sup_lexord sets Sup_measure'*
 ($\lambda U. \text{sigma } (\bigcup u \in U. \text{space } u) (\bigcup u \in U. \text{sets } u)$))
 ($\lambda U. \text{sigma } (\bigcup u \in U. \text{space } u) \{\}$)

definition *Inf_measure* :: 'a measure set \Rightarrow 'a measure **where**

Inf_measure $A = \text{Sup } \{x. \forall a \in A. x \leq a\}$

definition *inf_measure* :: 'a measure \Rightarrow 'a measure \Rightarrow 'a measure **where**

inf_measure $a \ b = \text{Inf } \{a, b\}$

definition *top_measure* :: 'a measure **where**

top_measure = *Inf* $\{\}$

end

7.4 Borel Space

theory *Borel_Space*

imports

Measurable Derivative Ordered_Euclidean_Space Extended_Real_Limits

begin

proposition *open_prod_generated*: *open* = *generate_topology* $\{A \times B \mid A \ B. \text{open } A \wedge \text{open } B\}$

proposition *mono_on_imp_deriv_nonneg*:

assumes *mono*: *mono_on* $A \ f$ **and** *deriv*: (*f has_real_derivative* D) (at x)

assumes $x \in \text{interior } A$

shows $D \geq 0$

proposition *mono_on_ctble_discont*:

fixes $f :: \text{real} \Rightarrow \text{real}$

fixes $A :: \text{real set}$

assumes *mono_on* $A \ f$

shows *countable* $\{a \in A. \neg \text{continuous (at } a \text{ within } A) f\}$

7.4.1 Generic Borel spaces

definition (in *topological_space*) *borel* :: 'a measure **where**

borel = *sigma UNIV* $\{S. \text{open } S\}$

theorem *second_countable_borel_measurable*:
fixes $X :: 'a::second_countable_topology\ set\ set$
assumes $eq: open = generate_topology\ X$
shows $borel = sigma\ UNIV\ X$

proposition *borel_eq_countable_basis*:
fixes $B :: 'a::topological_space\ set\ set$
assumes $countable\ B$
assumes $topological_basis\ B$
shows $borel = sigma\ UNIV\ B$

7.4.2 Borel spaces on order topologies

7.4.3 Borel spaces on topological monoids

7.4.4 Borel spaces on Euclidean spaces

7.4.5 Borel measurable operators

lemma *borel_measurable_complex_iff*:
 $f \in borel_measurable\ M \longleftrightarrow$
 $(\lambda x. Re\ (f\ x)) \in borel_measurable\ M \wedge (\lambda x. Im\ (f\ x)) \in borel_measurable\ M$
(is ?lhs \longleftrightarrow ?rhs)

7.4.6 Borel space on the extended reals

theorem *borel_measurable_ereal_iff_real*:
fixes $f :: 'a \Rightarrow ereal$
shows $f \in borel_measurable\ M \longleftrightarrow$
 $((\lambda x. real_of_ereal\ (f\ x)) \in borel_measurable\ M \wedge f - \{ \infty \} \cap space\ M \in sets\ M \wedge f - \{ -\infty \} \cap space\ M \in sets\ M)$

7.4.7 Borel space on the extended non-negative reals

definition [*simp*]: $is_borel\ f\ M \longleftrightarrow f \in borel_measurable\ M$

7.4.8 LIMSEQ is borel measurable

proposition *measurable_limit* [*measurable*]:
fixes $f :: nat \Rightarrow 'a \Rightarrow 'b::first_countable_topology$
assumes [*measurable*]: $\bigwedge n::nat. f\ n \in borel_measurable\ M$
shows $Measurable.pred\ M\ (\lambda x. (\lambda n. f\ n\ x) \longrightarrow c)$

end

7.5 Lebesgue Integration for Nonnegative Functions

theory *Nonnegative_Lebesgue_Integration*
imports *Measure_Space Borel_Space*
begin

7.5.1 Simple function

definition *simple_function* $M g \longleftrightarrow$
 $finite (g \text{ ' } space M) \wedge$
 $(\forall x \in g \text{ ' } space M. g - \{x\} \cap space M \in sets M)$

lemma *borel_measurable_implies_simple_function_sequence*:
fixes $u :: 'a \Rightarrow ennreal$
assumes $u[measurable]: u \in borel_measurable M$
shows $\exists f. incseq f \wedge (\forall i. (\forall x. f i x < top) \wedge simple_function M (f i)) \wedge u = (SUP i. f i)$

lemma *simple_function_induct*
 $[consumes 1, case_names\ cong\ set\ mult\ add, induct\ set: simple_function]$:
fixes $u :: 'a \Rightarrow ennreal$
assumes $u: simple_function M u$
assumes $cong: \bigwedge f g. simple_function M f \Longrightarrow simple_function M g \Longrightarrow (AE x$
in $M. f x = g x) \Longrightarrow P f \Longrightarrow P g$
assumes $set: \bigwedge A. A \in sets M \Longrightarrow P (indicator A)$
assumes $mult: \bigwedge u c. P u \Longrightarrow P (\lambda x. c * u x)$
assumes $add: \bigwedge u v. P u \Longrightarrow P v \Longrightarrow P (\lambda x. v x + u x)$
shows $P u$

lemma *borel_measurable_induct*
 $[consumes 1, case_names\ cong\ set\ mult\ add\ seq, induct\ set: borel_measurable]$:
fixes $u :: 'a \Rightarrow ennreal$
assumes $u: u \in borel_measurable M$
assumes $cong: \bigwedge f g. f \in borel_measurable M \Longrightarrow g \in borel_measurable M \Longrightarrow$
 $(\bigwedge x. x \in space M \Longrightarrow f x = g x) \Longrightarrow P g \Longrightarrow P f$
assumes $set: \bigwedge A. A \in sets M \Longrightarrow P (indicator A)$
assumes $mult': \bigwedge u c. c < top \Longrightarrow u \in borel_measurable M \Longrightarrow (\bigwedge x. x \in space$
 $M \Longrightarrow u x < top) \Longrightarrow P u \Longrightarrow P (\lambda x. c * u x)$
assumes $add: \bigwedge u v. u \in borel_measurable M \Longrightarrow (\bigwedge x. x \in space M \Longrightarrow u x <$
 $top) \Longrightarrow P u \Longrightarrow v \in borel_measurable M \Longrightarrow (\bigwedge x. x \in space M \Longrightarrow v x < top)$
 $\Longrightarrow (\bigwedge x. x \in space M \Longrightarrow u x = 0 \vee v x = 0) \Longrightarrow P v \Longrightarrow P (\lambda x. v x + u x)$
assumes $seq: \bigwedge U. (\bigwedge i. U i \in borel_measurable M) \Longrightarrow (\bigwedge i x. x \in space M \Longrightarrow$
 $U i x < top) \Longrightarrow (\bigwedge i. P (U i)) \Longrightarrow incseq U \Longrightarrow u = (SUP i. U i) \Longrightarrow P (SUP$
 $i. U i)$

shows $P u$

7.5.2 Simple integral

definition $simple_integral :: 'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal\ (integral^S)$
where

$$integral^S\ M\ f = (\sum x \in f\ 'space\ M. x * emeasure\ M\ (f\ -\ '\{x\} \cap space\ M))$$

7.5.3 Integral on nonnegative functions

definition $nn_integral :: 'a\ measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow ennreal\ (integral^N)$
where

$$integral^N\ M\ f = (SUP\ g \in \{g. simple_function\ M\ g \wedge g \leq f\}. integral^S\ M\ g)$$

theorem $nn_integral_monotone_convergence_SUP_AE$:

assumes $f: \bigwedge i. AE\ x\ in\ M. f\ i\ x \leq f\ (Suc\ i)\ x \wedge i. f\ i \in borel_measurable\ M$
shows $(\int^+ x. (SUP\ i. f\ i\ x)\ \partial M) = (SUP\ i. integral^N\ M\ (f\ i))$

theorem $nn_integral_suminf$:

assumes $f: \bigwedge i. f\ i \in borel_measurable\ M$
shows $(\int^+ x. (\sum i. f\ i\ x)\ \partial M) = (\sum i. integral^N\ M\ (f\ i))$

theorem $nn_integral_Markov_inequality$:

assumes $u: (\lambda x. u\ x * indicator\ A\ x) \in borel_measurable\ M$ **and** $A \in sets\ M$
shows $(emeasure\ M)\ (\{x \in A. 1 \leq c * u\ x\}) \leq c * (\int^+ x. u\ x * indicator\ A\ x\ \partial M)$
(is $(emeasure\ M)\ ?A \leq _ * ?PI)$

theorem $nn_integral_monotone_convergence_INF_AE$:

fixes $f :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes $f: \bigwedge i. AE\ x\ in\ M. f\ (Suc\ i)\ x \leq f\ i\ x$
and $[measurable]: \bigwedge i. f\ i \in borel_measurable\ M$
and $fin: (\int^+ x. f\ i\ x\ \partial M) < \infty$
shows $(\int^+ x. (INF\ i. f\ i\ x)\ \partial M) = (INF\ i. integral^N\ M\ (f\ i))$

theorem $nn_integral_liminf$:

fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes $u: \bigwedge i. u\ i \in borel_measurable\ M$
shows $(\int^+ x. liminf\ (\lambda n. u\ n\ x)\ \partial M) \leq liminf\ (\lambda n. integral^N\ M\ (u\ n))$

theorem $nn_integral_limsup$:

fixes $u :: nat \Rightarrow 'a \Rightarrow ennreal$
assumes $[measurable]: \bigwedge i. u\ i \in borel_measurable\ M\ w \in borel_measurable\ M$
assumes $bounds: \bigwedge i. AE\ x\ in\ M. u\ i\ x \leq w\ x$ **and** $w: (\int^+ x. w\ x\ \partial M) < \infty$
shows $limsup\ (\lambda n. integral^N\ M\ (u\ n)) \leq (\int^+ x. limsup\ (\lambda n. u\ n\ x)\ \partial M)$

theorem $nn_integral_dominated_convergence$:

assumes $[measurable]$:

$\bigwedge i. u\ i \in \text{borel_measurable } M\ u' \in \text{borel_measurable } M\ w \in \text{borel_measurable } M$
and *bound*: $\bigwedge j. \text{AE } x \text{ in } M. u\ j\ x \leq w\ x$
and *w*: $(\int^+ x. w\ x\ \partial M) < \infty$
and *u'*: $\text{AE } x \text{ in } M. (\lambda i. u\ i\ x) \longrightarrow u'\ x$
shows $(\lambda i. (\int^+ x. u\ i\ x\ \partial M)) \longrightarrow (\int^+ x. u'\ x\ \partial M)$

theorem *nn_integral_lfp*:

assumes *sets[simp]*: $\bigwedge s. \text{sets } (M\ s) = \text{sets } N$
assumes *f*: *sup_continuous* *f*
assumes *g*: *sup_continuous* *g*
assumes *meas*: $\bigwedge F. F \in \text{borel_measurable } N \implies f\ F \in \text{borel_measurable } N$
assumes *step*: $\bigwedge F\ s. F \in \text{borel_measurable } N \implies \text{integral}^N (M\ s)\ (f\ F) = g$
 $(\lambda s. \text{integral}^N (M\ s)\ F)\ s$
shows $(\int^+ \omega. \text{lfp } f\ \omega\ \partial M\ s) = \text{lfp } g\ s$

theorem *nn_integral_gfp*:

assumes *sets[simp]*: $\bigwedge s. \text{sets } (M\ s) = \text{sets } N$
assumes *f*: *inf_continuous* *f* **and** *g*: *inf_continuous* *g*
assumes *meas*: $\bigwedge F. F \in \text{borel_measurable } N \implies f\ F \in \text{borel_measurable } N$
assumes *bound*: $\bigwedge F\ s. F \in \text{borel_measurable } N \implies (\int^+ x. f\ F\ x\ \partial M\ s) < \infty$
assumes *non_zero*: $\bigwedge s. \text{emeasure } (M\ s)\ (\text{space } (M\ s)) \neq 0$
assumes *step*: $\bigwedge F\ s. F \in \text{borel_measurable } N \implies \text{integral}^N (M\ s)\ (f\ F) = g$
 $(\lambda s. \text{integral}^N (M\ s)\ F)\ s$
shows $(\int^+ \omega. \text{gfp } f\ \omega\ \partial M\ s) = \text{gfp } g\ s$

7.5.4 Integral under concrete measures

definition *density* :: 'a measure \Rightarrow ('a \Rightarrow ennreal) \Rightarrow 'a measure **where**
density *M* *f* = *measure_of* (*space* *M*) (*sets* *M*) $(\lambda A. \int^+ x. f\ x * \text{indicator } A\ x\ \partial M)$

lemma *nn_integral_density*:

assumes *f*: $f \in \text{borel_measurable } M$
assumes *g*: $g \in \text{borel_measurable } M$
shows $\text{integral}^N (\text{density } M\ f)\ g = (\int^+ x. f\ x * g\ x\ \partial M)$

definition *point_measure* :: 'a set \Rightarrow ('a \Rightarrow ennreal) \Rightarrow 'a measure **where**

point_measure *A* *f* = *density* (*count_space* *A*) *f*

definition *uniform_measure* *M* *A* = *density* *M* $(\lambda x. \text{indicator } A\ x / \text{emeasure } M\ A)$

definition *uniform_count_measure* *A* = *point_measure* *A* $(\lambda x. 1 / \text{card } A)$

end

7.6 Binary Product Measure

theory *Binary_Product_Measure*

imports *Nonnegative_Lebesgue_Integration*
begin

7.6.1 Binary products

definition *pair_measure* (**infixr** \otimes_M 80) **where**

$A \otimes_M B = \text{measure_of } (\text{space } A \times \text{space } B)$
 $\{a \times b \mid a \in \text{sets } A \wedge b \in \text{sets } B\}$
 $(\lambda X. \int^+ x. (\int^+ y. \text{indicator } X (x,y) \partial B) \partial A)$

proposition (**in** *sigma_finite_measure*) *emeasure_pair_measure_Times*:

assumes $A: A \in \text{sets } N$ **and** $B: B \in \text{sets } M$

shows $\text{emeasure } (N \otimes_M M) (A \times B) = \text{emeasure } N A * \text{emeasure } M B$

7.6.2 Binary products of σ -finite emeasure spaces

proposition (**in** *pair_sigma_finite*) *sigma_finite_up_in_pair_measure_generator*:

defines $E \equiv \{A \times B \mid A \in \text{sets } M1 \wedge B \in \text{sets } M2\}$

shows $\exists F::\text{nat} \Rightarrow ('a \times 'b) \text{ set. range } F \subseteq E \wedge \text{incseq } F \wedge (\bigcup i. F i) = \text{space } M1 \times \text{space } M2 \wedge$

$(\forall i. \text{emeasure } (M1 \otimes_M M2) (F i) \neq \infty)$

7.6.3 Fubini's theorem

proposition (**in** *pair_sigma_finite*) *nn_integral_snd*:

assumes $f[\text{measurable}]: f \in \text{borel_measurable } (M1 \otimes_M M2)$

shows $(\int^+ y. (\int^+ x. f (x, y) \partial M1) \partial M2) = \text{integral}^N (M1 \otimes_M M2) f$

theorem (**in** *pair_sigma_finite*) *Fubini*:

assumes $f: f \in \text{borel_measurable } (M1 \otimes_M M2)$

shows $(\int^+ y. (\int^+ x. f (x, y) \partial M1) \partial M2) = (\int^+ x. (\int^+ y. f (x, y) \partial M2) \partial M1)$

theorem (**in** *pair_sigma_finite*) *Fubini'*:

assumes $f: \text{case_prod } f \in \text{borel_measurable } (M1 \otimes_M M2)$

shows $(\int^+ y. (\int^+ x. f x y \partial M1) \partial M2) = (\int^+ x. (\int^+ y. f x y \partial M2) \partial M1)$

7.6.4 Products on counting spaces, densities and distributions

proposition *sigma_prod*:

assumes $X_cover: \exists E \subseteq A. \text{countable } E \wedge X = \bigcup E$ **and** $A: A \subseteq \text{Pow } X$

assumes $Y_cover: \exists E \subseteq B. \text{countable } E \wedge Y = \bigcup E$ **and** $B: B \subseteq \text{Pow } Y$

shows $\sigma X A \otimes_M \sigma Y B = \sigma (X \times Y) \{a \times b \mid a \in A \wedge b \in B\}$
(is $?P = ?S$)

proposition *sets_pair_eq*:

assumes $Ea: Ea \subseteq Pow \text{ (space } A) \text{ sets } A = \sigma_sets \text{ (space } A) Ea$
and $Ca: countable Ca Ca \subseteq Ea \cup Ca = \text{space } A$
and $Eb: Eb \subseteq Pow \text{ (space } B) \text{ sets } B = \sigma_sets \text{ (space } B) Eb$
and $Cb: countable Cb Cb \subseteq Eb \cup Cb = \text{space } B$
shows $\sigma (A \otimes_M B) = \sigma (\sigma \text{ (space } A \times \text{space } B) \{a \times b \mid a \in Ea \wedge b \in Eb\})$
(is $_ = \sigma (\sigma \ ?\Omega \ ?E)$)

proposition *borel_prod*:

$(borel \otimes_M borel) = (borel :: ('a::second_countable_topology \times 'b::second_countable_topology) \text{ measure})$
(is $?P = ?B$)

proposition *pair_measure_count_space*:

assumes $A: finite A$ **and** $B: finite B$
shows $count_space A \otimes_M count_space B = count_space (A \times B)$ **(is** $?P = ?C$)

theorem *pair_measure_density*:

assumes $f: f \in borel_measurable M1$
assumes $g: g \in borel_measurable M2$
assumes $\sigma_finite_measure M2 \sigma_finite_measure (density M2 g)$
shows $density M1 f \otimes_M density M2 g = density (M1 \otimes_M M2) (\lambda(x,y). f x * g y)$ **(is** $?L = ?R$)

proposition *nn_integral_fst_count_space*:

$(\int^+ x. \int^+ y. f(x, y) \partial count_space UNIV \partial count_space UNIV) = integral^N (count_space UNIV) f$
(is $?lhs = ?rhs$)

proposition *nn_integral_snd_count_space*:

$(\int^+ y. \int^+ x. f(x, y) \partial count_space UNIV \partial count_space UNIV) = integral^N (count_space UNIV) f$
(is $?lhs = ?rhs$)

7.6.5 Product of Borel spaces

theorem *borel_Times*:

fixes $A :: 'a::topological_space \text{ set}$ **and** $B :: 'b::topological_space \text{ set}$
assumes $A: A \in sets \ borel$ **and** $B: B \in sets \ borel$
shows $A \times B \in sets \ borel$

end

7.7 Finite Product Measure

theory *Finite_Product_Measure*
imports *Binary_Product_Measure Function_Topology*
begin

7.7.1 Finite product spaces

definition *prod_emb* **where**

$$\text{prod_emb } I M K X = (\lambda x. \text{restrict } x K) - ' X \cap (\prod_{E \ i \in I. \text{space } (M \ i)})$$

definition *PiM* :: *'i set* \Rightarrow (*'i* \Rightarrow *'a measure*) \Rightarrow (*'i* \Rightarrow *'a*) *measure* **where**

$$\text{PiM } I M = \text{extend_measure } (\prod_{E \ i \in I. \text{space } (M \ i)})$$

$$\{(J, X). (J \neq \{\} \vee I = \{\}) \wedge \text{finite } J \wedge J \subseteq I \wedge X \in (\prod_{j \in J. \text{sets } (M \ j)})\}$$

$$(\lambda(J, X). \text{prod_emb } I M J (\prod_{E \ j \in J. X \ j}))$$

$(\lambda(J, X). \prod_{j \in J \cup \{i \in I. \text{emeasure } (M \ i) (\text{space } (M \ i)) \neq 1\}. \text{if } j \in J \text{ then } \text{emeasure } (M \ j) (X \ j) \text{ else } \text{emeasure } (M \ j) (\text{space } (M \ j))})$

definition *prod_algebra* :: *'i set* \Rightarrow (*'i* \Rightarrow *'a measure*) \Rightarrow (*'i* \Rightarrow *'a*) *set set* **where**

$$\text{prod_algebra } I M = (\lambda(J, X). \text{prod_emb } I M J (\prod_{E \ j \in J. X \ j}) - '$$

$$\{(J, X). (J \neq \{\} \vee I = \{\}) \wedge \text{finite } J \wedge J \subseteq I \wedge X \in (\prod_{j \in J. \text{sets } (M \ j)})\}$$

proposition *prod_algebra_mono*:

$$\text{assumes } \text{space}: \bigwedge i. i \in I \implies \text{space } (E \ i) = \text{space } (F \ i)$$

$$\text{assumes } \text{sets}: \bigwedge i. i \in I \implies \text{sets } (E \ i) \subseteq \text{sets } (F \ i)$$

$$\text{shows } \text{prod_algebra } I E \subseteq \text{prod_algebra } I F$$

proposition *prod_algebra_cong*:

$$\text{assumes } I = J \text{ and } (\bigwedge i. i \in I \implies \text{sets } (M \ i) = \text{sets } (N \ i))$$

$$\text{shows } \text{prod_algebra } I M = \text{prod_algebra } J N$$

proposition *sets_PiM_single*: $\text{sets } (\text{PiM } I M) =$

$$\text{sigma_sets } (\prod_{E \ i \in I. \text{space } (M \ i)}) \{\{f \in \prod_{E \ i \in I. \text{space } (M \ i)}. f \ i \in A \mid i \ A. \ i \in I \wedge A \in \text{sets } (M \ i)\}$$

$$(\text{is } _ = \text{sigma_sets } \ ?\Omega \ ?R)$$

proposition *sets_PiM_sigma*:

$$\text{assumes } \Omega_cover: \bigwedge i. i \in I \implies \exists S \subseteq E \ i. \text{countable } S \wedge \Omega \ i = \bigcup S$$

$$\text{assumes } E: \bigwedge i. i \in I \implies E \ i \subseteq \text{Pow } (\Omega \ i)$$

$$\text{assumes } J: \bigwedge j. j \in J \implies \text{finite } j \cup J = I$$

$$\text{defines } P \equiv \{\{f \in (\prod_{E \ i \in I. \Omega \ i}). \forall i \in j. f \ i \in A \ i \mid A \ j. j \in J \wedge A \in \text{Pi } j \ E\}$$

$$\text{shows } \text{sets } (\prod_{M \ i \in I. \text{sigma } (\Omega \ i) (E \ i)}) = \text{sets } (\text{sigma } (\prod_{E \ i \in I. \Omega \ i) P)$$

proposition *measurable_PiM*:

$$\text{assumes } \text{space}: f \in \text{space } N \rightarrow (\prod_{E \ i \in I. \text{space } (M \ i)})$$

$$\text{assumes } \text{sets}: \bigwedge X \ J. J \neq \{\} \vee I = \{\} \implies \text{finite } J \implies J \subseteq I \implies (\bigwedge i. i \in J$$

$\implies X \ i \in \text{sets } (M \ i) \implies$
 $f \text{ -- 'prod_emb } I \ M \ J \ (Pi_E \ J \ X) \cap \text{space } N \in \text{sets } N$
shows $f \in \text{measurable } N \ (PiM \ I \ M)$

proposition *measurable_fun_upd*:

assumes $I: I = J \cup \{i\}$
assumes $f[\text{measurable}]: f \in \text{measurable } N \ (PiM \ J \ M)$
assumes $h[\text{measurable}]: h \in \text{measurable } N \ (M \ i)$
shows $(\lambda x. (f \ x) \ (i := h \ x)) \in \text{measurable } N \ (PiM \ I \ M)$

proposition *measure_eqI_PiM_finite*:

assumes $[\text{simp}]: \text{finite } I \ \text{sets } P = PiM \ I \ M \ \text{sets } Q = PiM \ I \ M$
assumes $\text{eq}: \bigwedge A. (\bigwedge i. i \in I \implies A \ i \in \text{sets } (M \ i)) \implies P \ (Pi_E \ I \ A) = Q \ (Pi_E \ I \ A)$
assumes $A: \text{range } A \subseteq \text{prod_algebra } I \ M \ (\bigcup i. A \ i) = \text{space } (PiM \ I \ M) \ \bigwedge i::\text{nat. } P \ (A \ i) \neq \infty$
shows $P = Q$

proposition *measure_eqI_PiM_infinite*:

assumes $[\text{simp}]: \text{sets } P = PiM \ I \ M \ \text{sets } Q = PiM \ I \ M$
assumes $\text{eq}: \bigwedge A \ J. \text{finite } J \implies J \subseteq I \implies (\bigwedge i. i \in J \implies A \ i \in \text{sets } (M \ i))$
 \implies
 $P \ (\text{prod_emb } I \ M \ J \ (Pi_E \ J \ A)) = Q \ (\text{prod_emb } I \ M \ J \ (Pi_E \ J \ A))$
assumes $A: \text{finite_measure } P$
shows $P = Q$

proposition (in *finite_product_sigma_finite*) *sigma_finite_pairs*:

$\exists F::'i \Rightarrow \text{nat} \Rightarrow 'a \ \text{set.}$
 $(\forall i \in I. \text{range } (F \ i) \subseteq \text{sets } (M \ i)) \wedge$
 $(\forall k. \forall i \in I. \text{emeasure } (M \ i) \ (F \ i \ k) \neq \infty) \wedge \text{incseq } (\lambda k. \Pi_E \ i \in I. F \ i \ k) \wedge$
 $(\bigcup k. \Pi_E \ i \in I. F \ i \ k) = \text{space } (PiM \ I \ M)$

lemma (in *product_sigma_finite*) *distr_merge*:

assumes $IJ[\text{simp}]: I \cap J = \{\}$ **and** $\text{fin}: \text{finite } I \ \text{finite } J$
shows $\text{distr } (PiM \ I \ M \ \otimes_M \ PiM \ J \ M) \ (PiM \ (I \cup J) \ M) \ (\text{merge } I \ J) = PiM \ (I \cup J) \ M$
(is ?D = ?P)

proposition (in *product_sigma_finite*) *product_nn_integral_fold*:

assumes $IJ: I \cap J = \{\}$ *finite* I *finite* J
and $f[\text{measurable}]: f \in \text{borel_measurable } (PiM \ (I \cup J) \ M)$
shows $\text{integral}^N \ (PiM \ (I \cup J) \ M) \ f = (\int^+ x. (\int^+ y. f \ (\text{merge } I \ J \ (x, y)) \ \partial(PiM \ J \ M)) \ \partial(PiM \ I \ M))$
(is ?lhs = ?rhs)

proposition (in *product_sigma_finite*) *product_nn_integral_insert*:

assumes $I[\text{simp}]: \text{finite } I \ i \notin I$
and $f: f \in \text{borel_measurable } (PiM \ (\text{insert } i \ I) \ M)$
shows $\text{integral}^N \ (PiM \ (\text{insert } i \ I) \ M) \ f = (\int^+ x. (\int^+ y. f \ (x(i := y)) \ \partial(M \ i))$

$\partial(Pi_M I M)$

proposition (in *product_sigma_finite*) *product_nn_integral_pair*:

assumes [*measurable*]: $case_prod\ f \in borel_measurable\ (M\ x \otimes_M M\ y)$

assumes *xy*: $x \neq y$

shows $(\int^{+\sigma}. f\ (\sigma\ x)\ (\sigma\ y)\ \partial Pi_M\ \{x, y\}\ M) = (\int^{+z}. f\ (fst\ z)\ (snd\ z)\ \partial(M\ x \otimes_M M\ y))$

7.7.2 Measurability

proposition *sets_PiM_equal_borel*:

$sets\ (Pi_M\ UNIV\ (\lambda i. ('a::countable). borel::('b::second_countable_topology\ measure))) = sets\ borel$

end

7.8 Caratheodory Extension Theorem

theory *Caratheodory*

imports *Measure_Space*

begin

7.8.1 Characterizations of Measures

definition *outer_measure_space* **where**

$outer_measure_space\ M\ f \iff positive\ M\ f \wedge increasing\ M\ f \wedge countably_subadditive\ M\ f$

Lambda Systems

definition *lambda_system* $:: 'a\ set \Rightarrow 'a\ set\ set \Rightarrow ('a\ set \Rightarrow ennreal) \Rightarrow 'a\ set\ set$

where

$lambda_system\ \Omega\ M\ f = \{l \in M. \forall x \in M. f\ (l \cap x) + f\ ((\Omega - l) \cap x) = f\ x\}$

proposition (in *sigma_algebra*) *lambda_system_caratheodory*:

assumes *oms*: *outer_measure_space* $M\ f$

and *A*: $range\ A \subseteq lambda_system\ \Omega\ M\ f$

and *disj*: *disjoint_family* A

shows $(\bigcup i. A\ i) \in lambda_system\ \Omega\ M\ f \wedge (\sum i. f\ (A\ i)) = f\ (\bigcup i. A\ i)$

proposition (in *sigma_algebra*) *caratheodory_lemma*:

assumes *oms*: *outer_measure_space* $M\ f$

defines $L \equiv lambda_system\ \Omega\ M\ f$

shows *measure_space* $\Omega\ L\ f$

definition *outer_measure* :: 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow 'a set \Rightarrow ennreal
where

outer_measure *M f X* =
 $(\text{INF } A \in \{A. \text{range } A \subseteq M \wedge \text{disjoint_family } A \wedge X \subseteq (\bigcup i. A \ i)\}. \sum i. f \ (A \ i))$

7.8.2 Caratheodory's theorem

theorem (in *ring_of_sets*) *caratheodory'*:

assumes *posf*: positive *M f* **and** *ca*: countably_additive *M f*
shows $\exists \mu :: 'a \text{ set} \Rightarrow \text{ennreal}. (\forall s \in M. \mu \ s = f \ s) \wedge \text{measure_space } \Omega$
(sigma_sets Ω *M*) μ

7.8.3 Volumes

definition *volume* :: 'a set set \Rightarrow ('a set \Rightarrow ennreal) \Rightarrow bool **where**

volume *M f* \longleftrightarrow
 $(f \ \{\} = 0) \wedge (\forall a \in M. 0 \leq f \ a) \wedge$
 $(\forall C \subseteq M. \text{disjoint } C \longrightarrow \text{finite } C \longrightarrow \bigcup C \in M \longrightarrow f \ (\bigcup C) = (\sum c \in C. f \ c))$

proposition *volume_finite_additive*:

assumes *volume* *M f*
assumes *A*: $\bigwedge i. i \in I \implies A \ i \in M$ *disjoint_family_on* *A I* *finite I* $\bigcup (A \ 'I) \in M$
shows $f \ (\bigcup (A \ 'I)) = (\sum i \in I. f \ (A \ i))$

proposition (in *semiring_of_sets*) *extend_volume*:

assumes *volume* *M* μ
shows $\exists \mu'. \text{volume_generated_ring } \mu' \wedge (\forall a \in M. \mu' \ a = \mu \ a)$

Caratheodory on semirings

theorem (in *semiring_of_sets*) *caratheodory*:

assumes *pos*: positive *M* μ **and** *ca*: countably_additive *M* μ
shows $\exists \mu' :: 'a \text{ set} \Rightarrow \text{ennreal}. (\forall s \in M. \mu' \ s = \mu \ s) \wedge \text{measure_space } \Omega$
(sigma_sets Ω *M*) μ'

proposition *extend_measure_caratheodory_pair*:

fixes *G* :: 'i \Rightarrow 'j \Rightarrow 'a set
assumes *M*: $M = \text{extend_measure } \Omega \ \{(a, b). P \ a \ b\} \ (\lambda(a, b). G \ a \ b) \ (\lambda(a, b). \mu \ a \ b)$
assumes *P i j*
assumes *semiring*: *semiring_of_sets* $\Omega \ \{G \ a \ b \mid a \ b. P \ a \ b\}$
assumes *empty*: $\bigwedge i \ j. P \ i \ j \implies G \ i \ j = \{\} \implies \mu \ i \ j = 0$
assumes *inj*: $\bigwedge i \ j \ k \ l. P \ i \ j \implies P \ k \ l \implies G \ i \ j = G \ k \ l \implies \mu \ i \ j = \mu \ k \ l$
assumes *nonneg*: $\bigwedge i \ j. P \ i \ j \implies 0 \leq \mu \ i \ j$
assumes *add*: $\bigwedge A :: \text{nat} \Rightarrow 'i. \bigwedge B :: \text{nat} \Rightarrow 'j. \bigwedge j \ k.$

$(\bigwedge n. P (A n) (B n)) \implies P j k \implies \text{disjoint_family } (\lambda n. G (A n) (B n)) \implies$
 $(\bigcup i. G (A i) (B i)) = G j k \implies (\sum n. \mu (A n) (B n)) = \mu j k$
shows $\text{emeasure } M (G i j) = \mu i j$

end

7.9 Bochner Integration for Vector-Valued Functions

theory *Bochner_Integration*

imports *Finite_Product_Measure*

beginproposition *borel_measurable_implies_sequence_metric:*

fixes $f :: 'a \Rightarrow 'b :: \{\text{metric_space, second_countable_topology}\}$

assumes [*measurable*]: $f \in \text{borel_measurable } M$

shows $\exists F. (\forall i. \text{simple_function } M (F i)) \wedge (\forall x \in \text{space } M. (\lambda i. F i x) \longrightarrow f x) \wedge$

$(\forall i. \forall x \in \text{space } M. \text{dist } (F i x) z \leq 2 * \text{dist } (f x) z)$

definition *simple_bochner_integral* :: $'a \text{ measure} \Rightarrow ('a \Rightarrow 'b :: \text{real_vector}) \Rightarrow 'b$
where

$\text{simple_bochner_integral } M f = (\sum y \in f' \text{space } M. \text{measure } M \{x \in \text{space } M. f x = y\} *_{\mathbb{R}} y)$

proposition *simple_bochner_integral_partition:*

assumes f : *simple_bochner_integrable* $M f$ **and** g : *simple_function* $M g$

assumes *sub*: $\bigwedge x y. x \in \text{space } M \implies y \in \text{space } M \implies g x = g y \implies f x = f y$

assumes v : $\bigwedge x. x \in \text{space } M \implies f x = v (g x)$

shows $\text{simple_bochner_integral } M f = (\sum y \in g' \text{space } M. \text{measure } M \{x \in \text{space } M. g x = y\} *_{\mathbb{R}} v y)$

(**is** $_ = ?r$)

proposition *has_bochner_integral_implies_finite_norm:*

$\text{has_bochner_integral } M f x \implies (\int^+ x. \text{norm } (f x) \partial M) < \infty$

proposition *has_bochner_integral_norm_bound:*

assumes i : *has_bochner_integral* $M f x$

shows $\text{norm } x \leq (\int^+ x. \text{norm } (f x) \partial M)$

definition *lebesgue_integral* (*integral^L*) **where**

$\text{integral}^L M f = (\text{if } \exists x. \text{has_bochner_integral } M f x \text{ then } \text{THE } x. \text{has_bochner_integral } M f x \text{ else } 0)$

proposition *nn_integral_dominated_convergence_norm:*

fixes $u' :: _ \Rightarrow _ :: \{\text{real_normed_vector, second_countable_topology}\}$

assumes [*measurable*]:

$\bigwedge i. u i \in \text{borel_measurable } M u' \in \text{borel_measurable } M w \in \text{borel_measurable } M$

and *bound*: $\bigwedge j. \text{AE } x \text{ in } M. \text{norm } (u j x) \leq w x$

and $w: (\int^+ x. w \ x \ \partial M) < \infty$
and $u': AE \ x \ in \ M. (\lambda i. u \ i \ x) \longrightarrow u' \ x$
shows $(\lambda i. (\int^+ x. norm \ (u' \ x - u \ i \ x) \ \partial M)) \longrightarrow 0$

proposition *integrableI_bounded*:

fixes $f :: 'a \Rightarrow 'b::\{banach, second_countable_topology\}$
assumes $f[measurable]: f \in borel_measurable \ M$ **and** $fin: (\int^+ x. norm \ (f \ x) \ \partial M) < \infty$
shows *integrable* $M \ f$

proposition *nn_integral_eq_integral*:

assumes $f: integrable \ M \ f$
assumes *nonneg*: $AE \ x \ in \ M. 0 \leq f \ x$
shows $(\int^+ x. f \ x \ \partial M) = integral^L \ M \ f$

proposition *integral_norm_bound [simp]*:

fixes $f :: _ \Rightarrow 'a :: \{banach, second_countable_topology\}$
shows $norm \ (integral^L \ M \ f) \leq (\int x. norm \ (f \ x) \ \partial M)$

proposition *integral_abs_bound [simp]*:

fixes $f :: 'a \Rightarrow real$ **shows** $abs \ (\int x. f \ x \ \partial M) \leq (\int x. |f \ x| \ \partial M)$

proposition *integrable_induct*[*consumes 1, case_names base add lim, induct pred: integrable*]:

fixes $f :: 'a \Rightarrow 'b::\{banach, second_countable_topology\}$
assumes *integrable* $M \ f$
assumes *base*: $\bigwedge A \ c. A \in sets \ M \Longrightarrow emeasure \ M \ A < \infty \Longrightarrow P \ (\lambda x. indicator \ A \ x \ *_{\mathbb{R}} \ c)$
assumes *add*: $\bigwedge f \ g. integrable \ M \ f \Longrightarrow P \ f \Longrightarrow integrable \ M \ g \Longrightarrow P \ g \Longrightarrow P \ (\lambda x. f \ x + g \ x)$
assumes *lim*: $\bigwedge f \ s. (\bigwedge i. integrable \ M \ (s \ i)) \Longrightarrow (\bigwedge i. P \ (s \ i)) \Longrightarrow (\bigwedge x. x \in space \ M \Longrightarrow (\lambda i. s \ i \ x) \longrightarrow f \ x) \Longrightarrow (\bigwedge i \ x. x \in space \ M \Longrightarrow norm \ (s \ i \ x) \leq 2 * norm \ (f \ x)) \Longrightarrow integrable \ M \ f \Longrightarrow P \ f$
shows $P \ f$

theorem *integral_Markov_inequality*:

assumes [*measurable*]: *integrable* $M \ u$ **and** $AE \ x \ in \ M. 0 \leq u \ x \ 0 < (c::real)$
shows $(emeasure \ M) \ \{x \in space \ M. u \ x \geq c\} \leq (1/c) * (\int x. u \ x \ \partial M)$

theorem *integral_Markov_inequality_measure*:

assumes [*measurable*]: *integrable* $M \ u$ **and** $A \in sets \ M$ **and** $AE \ x \ in \ M. 0 \leq u \ x \ 0 < (c::real)$
shows $measure \ M \ \{x \in space \ M. u \ x \geq c\} \leq (\int x. u \ x \ \partial M) / c$

theorem (*in finite_measure*) *second_moment_method*:

assumes [*measurable*]: $f \in M \rightarrow_M \ borel$
assumes *integrable* $M \ (\lambda x. f \ x \ ^2)$

```

defines  $\mu \equiv \text{lebesgue\_integral } M f$ 
assumes  $a > 0$ 
shows  $\text{measure } M \{x \in \text{space } M. |f x| \geq a\} \leq \text{lebesgue\_integral } M (\lambda x. f x ^ 2) / a^2$ 
proof –
  have  $\text{integrable: integrable } M f$ 
    using  $\text{assms by (blast dest: square\_integrable\_imp\_integrable)}$ 
  have  $\{x \in \text{space } M. |f x| \geq a\} = \{x \in \text{space } M. f x ^ 2 \geq a^2\}$ 
    using  $\langle a > 0 \rangle \text{ by (simp flip: abs\_le\_square\_iff)}$ 
  hence  $\text{measure } M \{x \in \text{space } M. |f x| \geq a\} = \text{measure } M \{x \in \text{space } M. f x ^ 2 \geq a^2\}$ 
    by simp
  also have  $\dots \leq \text{lebesgue\_integral } M (\lambda x. f x ^ 2) / a^2$ 
    using  $\text{assms by (intro integral\_Markov\_inequality\_measure) auto}$ 
  finally show  $?thesis .$ 
qed

```

proposition *tendsto_L1_int:*

```

fixes  $u :: \_ \Rightarrow \_ \Rightarrow 'b::\{\text{banach, second\_countable\_topology}\}$ 
assumes  $[\text{measurable}]: \bigwedge n. \text{integrable } M (u n) \text{ integrable } M f$ 
  and  $(\lambda n. (\int ^+ x. \text{norm}(u n x - f x) \partial M)) \longrightarrow 0) F$ 
shows  $(\lambda n. (\int x. u n x \partial M)) \longrightarrow (\int x. f x \partial M) F$ 

```

proposition *tendsto_L1_AE_subseq:*

```

fixes  $u :: \text{nat} \Rightarrow 'a \Rightarrow 'b::\{\text{banach, second\_countable\_topology}\}$ 
assumes  $[\text{measurable}]: \bigwedge n. \text{integrable } M (u n)$ 
  and  $(\lambda n. (\int x. \text{norm}(u n x) \partial M)) \longrightarrow 0$ 
shows  $\exists r::\text{nat} \Rightarrow \text{nat}. \text{strict\_mono } r \wedge (\text{AE } x \text{ in } M. (\lambda n. u (r n) x) \longrightarrow 0)$ 

```

7.9.1 Restricted measure spaces

7.9.2 Measure spaces with an associated density

7.9.3 Distributions

7.9.4 Lebesgue integration on *count_space*

7.9.5 Point measure

proposition *integrable_point_measure_finite:*

```

fixes  $g :: 'a \Rightarrow 'b::\{\text{banach, second\_countable\_topology}\}$  and  $f :: 'a \Rightarrow \text{real}$ 
assumes  $\text{finite } A$ 
shows  $\text{integrable (point\_measure } A f) g$ 

```

7.9.6 Lebesgue integration on *null_measure*

7.9.7 Legacy lemmas for the real-valued Lebesgue integral

theorem *real_lebesgue_integral_def:*

assumes $f[\text{measurable}]$: *integrable* M f
shows $\text{integral}^L M f = \text{enn2real} (\int^+ x. f x \partial M) - \text{enn2real} (\int^+ x. \text{ennreal} (- f x) \partial M)$

theorem *real_integrable_def*:

$\text{integrable } M f \longleftrightarrow f \in \text{borel_measurable } M \wedge$
 $(\int^+ x. \text{ennreal} (f x) \partial M) \neq \infty \wedge (\int^+ x. \text{ennreal} (- f x) \partial M) \neq \infty$

7.9.8 Product measure

proposition (*in sigma_finite_measure*) *borel_measurable_lebesgue_integral*[*measurable (raw)*]:

fixes $f :: _ \Rightarrow _ \Rightarrow _ :: \{\text{banach, second_countable_topology}\}$
assumes $f[\text{measurable}]$: *case_prod* $f \in \text{borel_measurable} (N \otimes_M M)$
shows $(\lambda x. \int y. f x y \partial M) \in \text{borel_measurable } N$

theorem (*in pair_sigma_finite*) *Fubini_integrable*:

fixes $f :: _ \Rightarrow _ :: \{\text{banach, second_countable_topology}\}$
assumes $f[\text{measurable}]$: $f \in \text{borel_measurable} (M1 \otimes_M M2)$
and *integ1*: *integrable* $M1$ $(\lambda x. \int y. \text{norm} (f (x, y)) \partial M2)$
and *integ2*: *AE* x *in* $M1$. *integrable* $M2$ $(\lambda y. f (x, y))$
shows *integrable* $(M1 \otimes_M M2)$ f

proposition (*in pair_sigma_finite*) *integral_fst'*:

fixes $f :: _ \Rightarrow _ :: \{\text{banach, second_countable_topology}\}$
assumes f : *integrable* $(M1 \otimes_M M2)$ f
shows $(\int x. (\int y. f (x, y) \partial M2) \partial M1) = \text{integral}^L (M1 \otimes_M M2) f$

proposition (*in pair_sigma_finite*) *Fubini_integral*:

fixes $f :: _ \Rightarrow _ \Rightarrow _ :: \{\text{banach, second_countable_topology}\}$
assumes f : *integrable* $(M1 \otimes_M M2)$ (*case_prod* f)
shows $(\int y. (\int x. f x y \partial M1) \partial M2) = (\int x. (\int y. f x y \partial M2) \partial M1)$

end

7.10 Complete Measures

theory *Complete_Measure*

imports *Bochner_Integration*

begin

locale *complete_measure* =

fixes $M :: 'a \text{ measure}$

assumes *complete*: $\bigwedge A B. B \subseteq A \implies A \in \text{null_sets } M \implies B \in \text{sets } M$

definition

split_completion $M A p = (\text{if } A \in \text{sets } M \text{ then } p = (A, \{\}) \text{ else$

$\exists N'. A = \text{fst } p \cup \text{snd } p \wedge \text{fst } p \cap \text{snd } p = \{\} \wedge \text{fst } p \in \text{sets } M \wedge \text{snd } p \subseteq N' \wedge N' \in \text{null_sets } M)$

definition

$\text{main_part } M A = \text{fst } (\text{Eps } (\text{split_completion } M A))$

definition

$\text{null_part } M A = \text{snd } (\text{Eps } (\text{split_completion } M A))$

definition $\text{completion} :: 'a \text{ measure} \Rightarrow 'a \text{ measure}$ **where**

$\text{completion } M = \text{measure_of } (\text{space } M) \{ S \cup N \mid S \cap N = \{\}. S \in \text{sets } M \wedge N' \in \text{null_sets } M \wedge N \subseteq N' \}$
 $(\text{emeasure } M \circ \text{main_part } M)$

lemma sets_completion :

$\text{sets } (\text{completion } M) = \{ S \cup N \mid S \cap N = \{\}. S \in \text{sets } M \wedge N' \in \text{null_sets } M \wedge N \subseteq N' \}$

lemma $\text{measurable_completion}$: $f \in M \rightarrow_M N \implies f \in \text{completion } M \rightarrow_M N$

lemma split_completion :

assumes $A \in \text{sets } (\text{completion } M)$

shows $\text{split_completion } M A (\text{main_part } M A, \text{null_part } M A)$

lemma $\text{emeasure_completion[simp]}$:

assumes $S: S \in \text{sets } (\text{completion } M)$

shows $\text{emeasure } (\text{completion } M) S = \text{emeasure } M (\text{main_part } M S)$

lemma $\text{completion_ex_borel_measurable}$:

fixes $g :: 'a \Rightarrow \text{ennreal}$

assumes $g: g \in \text{borel_measurable } (\text{completion } M)$

shows $\exists g' \in \text{borel_measurable } M. (\forall x \text{ in } M. g x = g' x)$

locale $\text{semifinite_measure} =$

fixes $M :: 'a \text{ measure}$

assumes semifinite :

$\bigwedge A. A \in \text{sets } M \implies \text{emeasure } M A = \infty \implies \exists B \in \text{sets } M. B \subseteq A \wedge \text{emeasure } M B < \infty$

locale $\text{locally_determined_measure} = \text{semifinite_measure} +$

assumes $\text{locally_determined}$:

$\bigwedge A. A \subseteq \text{space } M \implies (\bigwedge B. B \in \text{sets } M \implies \text{emeasure } M B < \infty \implies A \cap B \in \text{sets } M) \implies A \in \text{sets } M$

locale $\text{cld_measure} =$

$\text{complete_measure } M + \text{locally_determined_measure } M$ **for** $M :: 'a \text{ measure}$

definition $\text{outer_measure_of} :: 'a \text{ measure} \Rightarrow 'a \text{ set} \Rightarrow \text{ennreal}$

where $\text{outer_measure_of } M A = (\text{INF } B \in \{B \in \text{sets } M. A \subseteq B\}. \text{emeasure } M B)$

B)

definition *measurable_envelope* :: 'a measure \Rightarrow 'a set \Rightarrow 'a set \Rightarrow bool
where *measurable_envelope* M A E \longleftrightarrow
 $(A \subseteq E \wedge E \in \text{sets } M \wedge (\forall F \in \text{sets } M. \text{emeasure } M (F \cap E) = \text{outer_measure_of } M (F \cap A)))$

lemma *measurable_envelope_eq2*:
assumes $A \subseteq E$ $E \in \text{sets } M$ $\text{emeasure } M E < \infty$
shows *measurable_envelope* M A E \longleftrightarrow $(\text{emeasure } M E = \text{outer_measure_of } M A)$

proposition (in *complete_measure*) *fmeasurable_inner_outer*:
 $S \in \text{fmeasurable } M \longleftrightarrow$
 $(\forall e > 0. \exists T \in \text{fmeasurable } M. \exists U \in \text{fmeasurable } M. T \subseteq S \wedge S \subseteq U \wedge |\text{measure } M T - \text{measure } M U| < e)$
(is $_ \longleftrightarrow$ *?approx*)

end

7.11 Regularity of Measures

theory *Regularity*
imports *Measure_Space Borel_Space*
begin

theorem
fixes $M :: 'a :: \{\text{second_countable_topology}, \text{complete_space}\}$ *measure*
assumes $sb: \text{sets } M = \text{sets borel}$
assumes $\text{emeasure } M (\text{space } M) \neq \infty$
assumes $B \in \text{sets borel}$
shows *inner_regular*: $\text{emeasure } M B =$
 $(\text{SUP } K \in \{K. K \subseteq B \wedge \text{compact } K\}. \text{emeasure } M K)$ **(is** *?inner B*)
and *outer_regular*: $\text{emeasure } M B =$
 $(\text{INF } U \in \{U. B \subseteq U \wedge \text{open } U\}. \text{emeasure } M U)$ **(is** *?outer B*)

end

7.12 Lebesgue Measure

theory *Lebesgue_Measure*
imports
Finite_Product_Measure
Caratheodory
Complete_Measure
Summation_Tests
Regularity
begin

7.12.1 Measures defined by monotonous functions

definition *interval_measure* :: (real \Rightarrow real) \Rightarrow real measure **where**
interval_measure *F* =
 extend_measure UNIV $\{(a, b). a \leq b\}$ $(\lambda(a, b). \{a <..b\})$ $(\lambda(a, b). ennreal (F b - F a))$

lemma *emeasure_interval_measure_Ioc*:

assumes $a \leq b$

assumes *mono_F*: $\bigwedge x y. x \leq y \implies F x \leq F y$

assumes *right_cont_F* : $\bigwedge a. \text{continuous } (\text{at_right } a) F$

shows *emeasure* (*interval_measure* *F*) $\{a <..b\} = F b - F a$

lemma *sets_interval_measure* [*simp*, *measurable_cong*]:

sets (*interval_measure* *F*) = *sets borel*

lemma *sigma_finite_interval_measure*:

assumes *mono_F*: $\bigwedge x y. x \leq y \implies F x \leq F y$

assumes *right_cont_F* : $\bigwedge a. \text{continuous } (\text{at_right } a) F$

shows *sigma_finite_measure* (*interval_measure* *F*)

7.12.2 Lebesgue-Borel measure

definition *lborel* :: ('a :: euclidean_space) measure **where**

lborel = *distr* $(\prod_M b \in \text{Basis}. \text{interval_measure } (\lambda x. x))$ *borel* $(\lambda f. \sum b \in \text{Basis}. f b *_R b)$

abbreviation *lebesgue* :: 'a::euclidean_space measure

where *lebesgue* \equiv *completion lborel*

abbreviation *lebesgue_on* :: 'a set \Rightarrow 'a::euclidean_space measure

where *lebesgue_on* $\Omega \equiv$ *restrict_space* (*completion lborel*) Ω

7.12.3 Borel measurability

lemma *emeasure_lborel_cbox*[*simp*]:

assumes [*simp*]: $\bigwedge b. b \in \text{Basis} \implies l \cdot b \leq u \cdot b$

shows *emeasure* *lborel* (*cbox* *l u*) = $(\prod b \in \text{Basis}. (u - l) \cdot b)$

7.12.4 Affine transformation on the Lebesgue-Borel

lemma *lborel_eqI*:

fixes *M* :: 'a::euclidean_space measure

assumes *emeasure_eq*: $\bigwedge l u. (\bigwedge b. b \in \text{Basis} \implies l \cdot b \leq u \cdot b) \implies \text{emeasure } M$
 $(\text{box } l u) = (\prod_{b \in \text{Basis}} (u - l) \cdot b)$
assumes *sets_eq*: $\text{sets } M = \text{sets borel}$
shows $\text{lborel} = M$

lemma *lborel_affine_euclidean*:

fixes $c :: 'a :: \text{euclidean_space} \Rightarrow \text{real}$ **and** t
defines $T x \equiv t + (\sum_{j \in \text{Basis}} (c j * (x \cdot j)) *_R j)$
assumes $c: \bigwedge j. j \in \text{Basis} \implies c j \neq 0$
shows $\text{lborel} = \text{density } (\text{distr lborel borel } T) (\lambda_. (\prod_{j \in \text{Basis}} |c j|))$ (**is** $_ = ?D$)

lemma *lborel_integral_real_affine*:

fixes $f :: \text{real} \Rightarrow 'a :: \{\text{banach}, \text{second_countable_topology}\}$ **and** $c :: \text{real}$
assumes $c: c \neq 0$ **shows** $(\int x. f x \partial \text{lborel}) = |c| *_R (\int x. f (t + c * x) \partial \text{lborel})$

corollary *lebesgue_real_affine*:

$c \neq 0 \implies \text{lebesgue} = \text{density } (\text{distr lebesgue lebesgue } (\lambda x. t + c * x)) (\lambda_. \text{ennreal } (\text{abs } c))$

lemma *lborel_prod*:

$\text{lborel} \otimes_M \text{lborel} = (\text{lborel} :: ('a :: \text{euclidean_space} \times 'b :: \text{euclidean_space}) \text{ measure})$

7.12.5 Lebesgue measurable sets

abbreviation *lmeasurable* :: $'a :: \text{euclidean_space}$ *set set*

where

$\text{lmeasurable} \equiv \text{fmeasurable lebesgue}$

lemma *lmeasurable_iff_integrable*:

$S \in \text{lmeasurable} \iff \text{integrable lebesgue } (\text{indicator } S :: 'a :: \text{euclidean_space} \Rightarrow \text{real})$

7.12.6 A nice lemma for negligibility proofs

proposition *starlike_negligible_bounded_gmeasurable*:

fixes $S :: 'a :: \text{euclidean_space}$ *set*

assumes $S: S \in \text{sets lebesgue}$ **and** *bounded* S

and *eq1*: $\bigwedge c x. \llbracket (c *_R x) \in S; 0 \leq c; x \in S \rrbracket \implies c = 1$

shows $S \in \text{null_sets lebesgue}$

corollary *starlike_negligible_compact*:

$\text{compact } S \implies (\bigwedge c x. \llbracket (c *_R x) \in S; 0 \leq c; x \in S \rrbracket \implies c = 1) \implies S \in \text{null_sets lebesgue}$

proposition *outer_regular_lborel_le*:

assumes $B[\text{measurable}]$: $B \in \text{sets borel}$ **and** $0 < (e::\text{real})$
obtains U **where** $\text{open } U \ B \subseteq U$ **and** $\text{emeasure lborel } (U - B) \leq e$

lemma *outer_regular_lborel*:

assumes B : $B \in \text{sets borel}$ **and** $0 < (e::\text{real})$
obtains U **where** $\text{open } U \ B \subseteq U$ $\text{emeasure lborel } (U - B) < e$

7.12.7 F _sigma and G _delta sets.

inductive *fsigma* :: $'a::\text{topological_space set} \Rightarrow \text{bool}$ **where**
 $(\bigwedge n::\text{nat. closed } (F\ n)) \Longrightarrow \text{fsigma } (\bigcup (F\ ' \text{UNIV}))$

inductive *gdelta* :: $'a::\text{topological_space set} \Rightarrow \text{bool}$ **where**
 $(\bigwedge n::\text{nat. open } (F\ n)) \Longrightarrow \text{gdelta } (\bigcap (F\ ' \text{UNIV}))$

end

7.13 Tagged Divisions for Henstock-Kurzweil Integration

theory *Tagged_Division*
imports *Topology_Euclidean_Space*
begin

7.13.1 Some useful lemmas about intervals

7.13.2 Bounds on intervals where they exist

definition *interval_upperbound* :: $('a::\text{euclidean_space}) \text{ set} \Rightarrow 'a$
where $\text{interval_upperbound } s = (\sum i \in \text{Basis. } (\text{SUP } x \in s. x \cdot i) *_{\mathbb{R}} i)$

definition *interval_lowerbound* :: $('a::\text{euclidean_space}) \text{ set} \Rightarrow 'a$
where $\text{interval_lowerbound } s = (\sum i \in \text{Basis. } (\text{INF } x \in s. x \cdot i) *_{\mathbb{R}} i)$

7.13.3 The notion of a gauge — simply an open set containing the point

definition *gauge* $\gamma \longleftrightarrow (\forall x. x \in \gamma \ x \wedge \text{open } (\gamma\ x))$

7.13.4 Attempt a systematic general set of "offset" results for components

7.13.5 Divisions

definition *division_of* (**infixl** *division'_of* 40)

where

$$\begin{aligned}
 s \text{ division_of } i &\longleftrightarrow \\
 &\text{finite } s \wedge \\
 &(\forall K \in s. K \subseteq i \wedge K \neq \{\}) \wedge (\exists a b. K = \text{cbox } a b) \wedge \\
 &(\forall K1 \in s. \forall K2 \in s. K1 \neq K2 \longrightarrow \text{interior}(K1) \cap \text{interior}(K2) = \{\}) \wedge \\
 &(\bigcup s = i)
 \end{aligned}$$

proposition *partial_division_extend_interval:*

assumes $p \text{ division_of } (\bigcup p) (\bigcup p) \subseteq \text{cbox } a b$
obtains $q \text{ where } p \subseteq q \text{ } q \text{ division_of } \text{cbox } a b \text{ } (b::'a::\text{euclidean_space})$

proposition *division_union_intervals_exists:*

assumes $\text{cbox } a b \neq \{\}$
obtains $p \text{ where } (\text{insert } (\text{cbox } a b) p) \text{ division_of } (\text{cbox } a b \cup \text{cbox } c d)$

7.13.6 Tagged (partial) divisions

definition *tagged_partial_division_of (infixr tagged'_partial'_division'_of 40)*

where $s \text{ tagged_partial_division_of } i \longleftrightarrow$
 $\text{finite } s \wedge$
 $(\forall x K. (x, K) \in s \longrightarrow x \in K \wedge K \subseteq i \wedge (\exists a b. K = \text{cbox } a b)) \wedge$
 $(\forall x1 K1 x2 K2. (x1, K1) \in s \wedge (x2, K2) \in s \wedge (x1, K1) \neq (x2, K2) \longrightarrow$
 $\text{interior } K1 \cap \text{interior } K2 = \{\})$

definition *tagged_division_of (infixr tagged'_division'_of 40)*

where $s \text{ tagged_division_of } i \longleftrightarrow s \text{ tagged_partial_division_of } i \wedge (\bigcup \{K. \exists x. (x, K) \in s\} = i)$

7.13.7 Functions closed on boxes: morphisms from boxes to monoids

Using additivity of lifted function to encode definedness. **definition**

lift_option :: ('a \Rightarrow 'b \Rightarrow 'c) \Rightarrow 'a option \Rightarrow 'b option \Rightarrow 'c option

where

lift_option f a' b' = Option.bind a' ($\lambda a. \text{Option.bind } b' (\lambda b. \text{Some } (f a b))$)

lemma *comm_monoid_lift_option:*

assumes *comm_monoid f z*
shows *comm_monoid (lift_option f) (Some z)*

Misc

Division points **definition** *division_points (k::('a::euclidean_space) set) d =*

$$\{(j,x). j \in \text{Basis} \wedge (\text{interval_lowerbound } k) \cdot j < x \wedge x < (\text{interval_upperbound } k) \cdot j \wedge (\exists i \in d. (\text{interval_lowerbound } i) \cdot j = x \vee (\text{interval_upperbound } i) \cdot j = x)\}$$

Operative

proposition *tagged_division*:

assumes $d \text{ tagged_division_of } (\text{cbox } a \ b)$
shows $F (\lambda(_, l). g \ l) \ d = g (\text{cbox } a \ b)$

7.13.8 Special case of additivity we need for the FTC

7.13.9 Fine-ness of a partition w.r.t. a gauge

definition *fine* (infix *fine* 46)

where $d \text{ fine } s \longleftrightarrow (\forall (x,k) \in s. k \subseteq d \ x)$

7.13.10 Some basic combining lemmas

7.13.11 General bisection principle for intervals; might be useful elsewhere

7.13.12 Cousin's lemma

7.13.13 A technical lemma about "refinement" of division

Covering lemma

proposition *covering_lemma*:

assumes $S \subseteq \text{cbox } a \ b \ \text{box } a \ b \neq \{\}$ *gauge* g

obtains \mathcal{D} **where**

countable $\mathcal{D} \ \bigcup \mathcal{D} \subseteq \text{cbox } a \ b$

$\bigwedge K. K \in \mathcal{D} \implies \text{interior } K \neq \{\} \wedge (\exists c \ d. K = \text{cbox } c \ d)$

pairwise $(\lambda A \ B. \text{interior } A \cap \text{interior } B = \{\}) \ \mathcal{D}$

$\bigwedge K. K \in \mathcal{D} \implies \exists x \in S \cap K. K \subseteq g \ x$

$\bigwedge u \ v. \text{cbox } u \ v \in \mathcal{D} \implies \exists n. \forall i \in \text{Basis}. v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2^n$
 $S \subseteq \bigcup \mathcal{D}$

7.13.14 Division filter

definition *division_filter* :: $'a::\text{euclidean_space set} \Rightarrow ('a \times 'a \ \text{set}) \ \text{set filter}$

where $\text{division_filter } s = (\text{INF } g \in \{g. \text{gauge } g\}. \text{principal } \{p. p \text{ tagged_division_of } s \wedge g \ \text{fine } p\})$

proposition *eventually_division_filter*:

$(\forall_F p \ \text{in } \text{division_filter } s. P \ p) \longleftrightarrow$

$(\exists g. \text{gauge } g \wedge (\forall p. p \text{ tagged_division_of } s \wedge g \ \text{fine } p \implies P \ p))$

end

7.14 Henstock-Kurzweil Gauge Integration in Many Dimensions

theory *Henstock_Kurzweil_Integration*
imports
Lebesgue_Measure Tagged_Division
begin

7.14.1 Content (length, area, volume...) of an interval

7.14.2 Gauge integral

7.14.3 Basic theorems about integrals

corollary *integral_mult_left* [*simp*]:
fixes $c :: 'a :: \{\text{real_normed_algebra}, \text{division_ring}\}$
shows $\text{integral } S (\lambda x. f x * c) = \text{integral } S f * c$

corollary *integral_mult_right* [*simp*]:
fixes $c :: 'a :: \{\text{real_normed_field}\}$
shows $\text{integral } S (\lambda x. c * f x) = c * \text{integral } S f$

corollary *integral_divide* [*simp*]:
fixes $z :: 'a :: \text{real_normed_field}$
shows $\text{integral } S (\lambda x. f x / z) = \text{integral } S (\lambda x. f x) / z$

7.14.4 Cauchy-type criterion for integrability

proposition *integrable_Cauchy*:
fixes $f :: 'n :: \text{euclidean_space} \Rightarrow 'a :: \{\text{real_normed_vector}, \text{complete_space}\}$
shows $f \text{ integrable_on } \text{cbox } a \ b \iff$
 $(\forall e > 0. \exists \gamma. \text{gauge } \gamma \wedge$
 $(\forall \mathcal{D}1 \ \mathcal{D}2. \mathcal{D}1 \text{ tagged_division_of } (\text{cbox } a \ b) \wedge \gamma \text{ fine } \mathcal{D}1 \wedge$
 $\mathcal{D}2 \text{ tagged_division_of } (\text{cbox } a \ b) \wedge \gamma \text{ fine } \mathcal{D}2 \longrightarrow$
 $\text{norm } ((\sum (x, K) \in \mathcal{D}1. \text{content } K *_{\mathbb{R}} f x) - (\sum (x, K) \in \mathcal{D}2. \text{content } K *_{\mathbb{R}}$
 $f x)) < e))$
(is ?l = ($\forall e > 0. \exists \gamma. ?P \ e \ \gamma$))

7.14.5 Additivity of integral on abutting intervals

proposition *has_integral_split*:
fixes $f :: 'a :: \text{euclidean_space} \Rightarrow 'b :: \text{real_normed_vector}$

assumes $fi: (f \text{ has_integral } i) (cbox \ a \ b \cap \{x. x \cdot k \leq c\})$
and $fj: (f \text{ has_integral } j) (cbox \ a \ b \cap \{x. x \cdot k \geq c\})$
and $k: k \in \text{Basis}$
shows $(f \text{ has_integral } (i + j)) (cbox \ a \ b)$

7.14.6 A sort of converse, integrability on subintervals

7.14.7 Bounds on the norm of Riemann sums and the integral itself

corollary *integrable_bound*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{real_normed_vector}$
assumes $0 \leq B$
and $f \text{ integrable_on } (cbox \ a \ b)$
and $\bigwedge x. x \in cbox \ a \ b \implies \text{norm } (f \ x) \leq B$
shows $\text{norm } (\text{integral } (cbox \ a \ b) \ f) \leq B * \text{content } (cbox \ a \ b)$

7.14.8 Similar theorems about relationship among components

7.14.9 Uniform limit of integrable functions is integrable

7.14.10 Negligible sets

proposition *negligible_standard_hyperplane[intro]*:

fixes $k :: 'a::\text{euclidean_space}$
assumes $k: k \in \text{Basis}$
shows $\text{negligible } \{x. x \cdot k = c\}$

corollary *negligible_standard_hyperplane_cart*:

fixes $k :: 'a::\text{finite}$
shows $\text{negligible } \{x. x \$k = (0::\text{real})\}$

proposition *has_integral_negligible*:

fixes $f :: 'b::\text{euclidean_space} \Rightarrow 'a::\text{real_normed_vector}$
assumes $\text{negs: negligible } S$
and $\bigwedge x. x \in (T - S) \implies f \ x = 0$
shows $(f \text{ has_integral } 0) \ T$

7.14.11 Some other trivialities about negligible sets

7.14.12 Finite case of the spike theorem is quite commonly needed

corollary *has_integral_bound_real*:

fixes $f :: \text{real} \Rightarrow 'b::\text{real_normed_vector}$

assumes $0 \leq B$ *finite* S

and $(f \text{ has_integral } i) \{a..b\}$

and $\bigwedge x. x \in \{a..b\} - S \implies \text{norm } (f x) \leq B$

shows $\text{norm } i \leq B * \text{content } \{a..b\}$

7.14.13 In particular, the boundary of an interval is negligible

7.14.14 Integrability of continuous functions

7.14.15 Specialization of additivity to one dimension

7.14.16 A useful lemma allowing us to factor out the content size

7.14.17 Fundamental theorem of calculus

theorem *fundamental_theorem_of_calculus*:

fixes $f :: \text{real} \Rightarrow 'a::\text{banach}$

assumes $a \leq b$

and $\text{vecd}: \bigwedge x. x \in \{a..b\} \implies (f \text{ has_vector_derivative } f' x) \text{ (at } x \text{ within } \{a..b\})$

shows $(f' \text{ has_integral } (f b - f a)) \{a..b\}$

7.14.18 Taylor series expansion

7.14.19 Only need trivial subintervals if the interval itself is trivial

proposition *division_of_nontrivial*:

fixes $\mathcal{D} :: 'a::\text{euclidean_space}$ *set set*

assumes $\text{sdiv}: \mathcal{D} \text{ division_of } (\text{cbox } a \ b)$

and $\text{cont0}: \text{content } (\text{cbox } a \ b) \neq 0$

shows $\{k. k \in \mathcal{D} \wedge \text{content } k \neq 0\} \text{ division_of } (\text{cbox } a \ b)$

- 7.14.20 Integrability on subintervals
- 7.14.21 Combining adjacent intervals in 1 dimension
- 7.14.22 Reduce integrability to "local" integrability
- 7.14.23 Second FTC or existence of antiderivative

- 7.14.24 Combined fundamental theorem of calculus
- 7.14.25 General "twiddling" for interval-to-interval function image
- 7.14.26 Special case of a basic affine transformation
- 7.14.27 Special case of stretching coordinate axes separately
- 7.14.28 even more special cases
- 7.14.29 Stronger form of FCT; quite a tedious proof

theorem *fundamental_theorem_of_calculus_interior*:

fixes $f :: \text{real} \Rightarrow 'a::\text{real_normed_vector}$
assumes $a \leq b$
and *contf*: *continuous_on* $\{a..b\}$ f
and *derf*: $\bigwedge x. x \in \{a <..< b\} \implies (f \text{ has_vector_derivative } f' x)$ (at x)
shows $(f' \text{ has_integral } (f b - f a)) \{a..b\}$

- 7.14.30 Stronger form with finite number of exceptional points

corollary *fundamental_theorem_of_calculus_strong*:

fixes $f :: \text{real} \Rightarrow 'a::\text{banach}$
assumes *finite* S
and $a \leq b$
and *vec*: $\bigwedge x. x \in \{a..b\} - S \implies (f \text{ has_vector_derivative } f'(x))$ (at x)
and *continuous_on* $\{a..b\}$ f
shows $(f' \text{ has_integral } (f b - f a)) \{a..b\}$

proposition *indefinite_integral_continuous_left*:

fixes $f :: \text{real} \Rightarrow 'a::\text{banach}$
assumes *intf*: f *integrable_on* $\{a..b\}$ **and** $a < c \leq b$ $e > 0$
obtains d **where** $d > 0$
and $\forall t. c - d < t \wedge t \leq c \implies \text{norm } (\text{integral } \{a..c\} f - \text{integral } \{a..t\} f) < e$

theorem *integral_has_vector_derivative'*:

fixes $f :: \text{real} \Rightarrow 'b::\text{banach}$
assumes $\text{continuous_on } \{a..b\} f$
and $x \in \{a..b\}$
shows $((\lambda u. \text{integral } \{u..b\} f) \text{ has_vector_derivative } - f x) \text{ (at } x \text{ within } \{a..b\})$

7.14.31 This doesn't directly involve integration, but that gives an easy proof

7.14.32 Generalize a bit to any convex set

7.14.33 Integrating characteristic function of an interval

corollary $\text{has_integral_restrict_UNIV}$:

fixes $f :: 'n::\text{euclidean_space} \Rightarrow 'a::\text{banach}$
shows $((\lambda x. \text{if } x \in s \text{ then } f x \text{ else } 0) \text{ has_integral } i) \text{ UNIV} \iff (f \text{ has_integral } i) s$

7.14.34 Integrals on set differences

corollary $\text{integral_spike_set}$:

fixes $f :: 'n::\text{euclidean_space} \Rightarrow 'a::\text{banach}$
assumes $\text{negligible } \{x \in S - T. f x \neq 0\} \text{ negligible } \{x \in T - S. f x \neq 0\}$
shows $\text{integral } S f = \text{integral } T f$

7.14.35 More lemmas that are useful later

7.14.36 Continuity of the integral (for a 1-dimensional interval)

7.14.37 A straddling criterion for integrability

7.14.38 Adding integrals over several sets

7.14.39 Also tagged divisions

7.14.40 Henstock's lemma

7.14.41 Monotone convergence (bounded interval first)

- 7.14.42 differentiation under the integral sign
- 7.14.43 Exchange uniform limit and integral
- 7.14.44 Integration by parts
- 7.14.45 Integration by substitution
- 7.14.46 Compute a double integral using iterated integrals and switching the order of integration

theorem *integral_swap_continuous:*

fixes $f :: ['a::euclidean_space, 'b::euclidean_space] \Rightarrow 'c::banach$

assumes *continuous_on* (cbox (a,c) (b,d)) ($\lambda(x,y). f\ x\ y$)

shows $integral\ (cbox\ a\ b)\ (\lambda x. integral\ (cbox\ c\ d)\ (f\ x)) =$
 $integral\ (cbox\ c\ d)\ (\lambda y. integral\ (cbox\ a\ b)\ (\lambda x. f\ x\ y))$

- 7.14.47 Definite integrals for exponential and power function

end

Chapter 8

Kronecker's Theorem with Applications

theory *Kronecker_Approximation_Theorem*

imports *Complex_Transcendental_Henstock_Kurzweil_Integration*
HOL-Real_Asymp.Real_Asymp

begin

8.1 Dirichlet's Approximation Theorem

theorem *Dirichlet_approx_simult:*

fixes $\vartheta :: \text{nat} \Rightarrow \text{real}$ **and** $N\ n :: \text{nat}$

assumes $N > 0$

obtains $q\ p$ **where** $0 < q \leq \text{int } (N^n)$

and $\bigwedge i. i < n \implies |\text{of_int } q * \vartheta\ i - \text{of_int}(p\ i)| < 1/N$

corollary *Dirichlet_approx:*

fixes $\vartheta :: \text{real}$ **and** $N :: \text{nat}$

assumes $N > 0$

obtains $h\ k$ **where** $0 < k \leq \text{int } N$ $|\text{of_int } k * \vartheta - \text{of_int } h| < 1/N$

corollary *Dirichlet_approx_coprime:*

fixes $\vartheta :: \text{real}$ **and** $N :: \text{nat}$

assumes $N > 0$

obtains $h\ k$ **where** $\text{coprime } h\ k$ $0 < k \leq \text{int } N$ $|\text{of_int } k * \vartheta - \text{of_int } h| < 1/N$

theorem *infinite_approx_set:*

assumes *infinite* (*approx_set* ϑ)

shows $\exists h\ k. (h, k) \in \text{approx_set } \vartheta \wedge k > K$

theorem *rational_iff_finite_approx_set:*

shows $\vartheta \in \mathbb{Q} \iff \text{finite } (\text{approx_set } \vartheta)$

8.2 Kronecker's Approximation Theorem: the One-dimensional Case

theorem *Kronecker_approx_1_explicit*:

fixes $\vartheta :: \text{real}$

assumes $\vartheta \notin \mathbb{Q}$ **and** $\alpha: 0 \leq \alpha \leq 1$ **and** $\varepsilon > 0$

obtains k **where** $k > 0$ $|\text{frac}(\text{real } k * \vartheta) - \alpha| < \varepsilon$

corollary *Kronecker_approx_1*:

fixes $\vartheta :: \text{real}$

assumes $\vartheta \notin \mathbb{Q}$

shows $\text{closure}(\text{range}(\lambda n. \text{frac}(\text{real } n * \vartheta))) = \{0..1\}$ (**is** $?C = _$)

corollary *sequence_of_fractional_parts_is_dense*:

fixes $\vartheta :: \text{real}$

assumes $\vartheta \notin \mathbb{Q}$ $\varepsilon > 0$

obtains $h k$ **where** $k > 0$ $|\text{of_int } k * \vartheta - \text{of_int } h - \alpha| < \varepsilon$

8.3 Extension of Kronecker's Theorem to Simultaneous Approximation

8.3.1 Towards Lemma 1

8.3.2 Towards Lemma 2

8.3.3 Towards lemma 3

8.3.4 And finally Kroncker's theorem itself

theorem *Kronecker_thm_1*:

fixes $\alpha \vartheta :: \text{nat} \Rightarrow \text{real}$ **and** $n :: \text{nat}$

assumes *indp*: *module.independent* $(\lambda r. (*) (\text{real_of_int } r)) (\vartheta \text{ ' } \{..<n\})$

and *inj* ϑ : *inj_on* $\vartheta \{..<n\}$ **and** $\varepsilon > 0$

obtains $t h$ **where** $\bigwedge i. i < n \implies |t * \vartheta i - \text{of_int } (h i) - \alpha i| < \varepsilon$

corollary *Kronecker_thm_2*:

fixes $\alpha \vartheta :: \text{nat} \Rightarrow \text{real}$ **and** $n :: \text{nat}$

assumes *indp*: *module.independent* $(\lambda r x. \text{of_int } r * x) (\vartheta \text{ ' } \{..n\})$

and *inj* ϑ : *inj_on* $\vartheta \{..n\}$ **and** [*simp*]: $\vartheta n = 1$ **and** $\varepsilon > 0$

obtains $k m$ **where** $\bigwedge i. i < n \implies |\text{of_int } k * \vartheta i - \text{of_int } (m i) - \alpha i| < \varepsilon$

end

8.4 Bernstein-Weierstrass and Stone-Weierstrass

```
theory Weierstrass_Theorems
imports Uniform_Limit Path_Connected Derivative
begin
```

8.4.1 Bernstein polynomials

```
definition Bernstein :: [nat,nat,real] => real where
  Bernstein n k x ≡ of_nat (n choose k) * x^k * (1 - x)^(n - k)
```

8.4.2 Explicit Bernstein version of the 1D Weierstrass approximation theorem

```
theorem Bernstein_Weierstrass:
  fixes f :: real => real
  assumes contf: continuous_on {0..1} f and e: 0 < e
  shows ∃ N. ∀ n x. N ≤ n ∧ x ∈ {0..1}
    → |f x - (∑ k≤n. f(k/n) * Bernstein n k x)| < e
```

8.4.3 General Stone-Weierstrass theorem

```
definition normf :: ('a::t2_space => real) => real
  where normf f ≡ SUP x∈S. |f x|
proposition (in function_ring_on) Stone_Weierstrass_basic:
  assumes f: continuous_on S f and e: e > 0
  shows ∃ g ∈ R. ∀ x∈S. |f x - g x| < e
```

```
theorem (in function_ring_on) Stone_Weierstrass:
  assumes f: continuous_on S f
  shows ∃ F ∈ UNIV → R. LIM n sequentially. F n :> uniformly_on S f
```

```
corollary Stone_Weierstrass_HOL:
```

```
  fixes R :: ('a::t2_space => real) set and S :: 'a set
  assumes compact S ∧ c. P(λx. c::real)
    ∧ f. P f ⇒ continuous_on S f
    ∧ f g. P(f) ∧ P(g) ⇒ P(λx. f x + g x) ∧ f g. P(f) ∧ P(g) ⇒ P(λx. f
x * g x)
    ∧ x y. x ∈ S ∧ y ∈ S ∧ x ≠ y ⇒ ∃ f. P(f) ∧ f x ≠ f y
    continuous_on S f
    0 < e
  shows ∃ g. P(g) ∧ (∀ x ∈ S. |f x - g x| < e)
```

8.4.4 Polynomial functions

definition *polynomial_function* :: ('a::real_normed_vector \Rightarrow 'b::real_normed_vector) \Rightarrow bool
where
polynomial_function p \equiv (\forall f. bounded_linear f \longrightarrow real_polynomial_function (f o p))

8.4.5 Stone-Weierstrass theorem for polynomial functions

theorem *Stone_Weierstrass_polynomial_function*:
fixes f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space
assumes S: compact S
and f: continuous_on S f
and e: 0 < e
shows \exists g. polynomial_function g \wedge (\forall x \in S. norm(f x - g x) < e)

proposition *Stone_Weierstrass_uniform_limit*:
fixes f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space
assumes S: compact S
and f: continuous_on S f
obtains g **where** uniform_limit S g f sequentially \wedge n. polynomial_function (g n)

8.4.6 Polynomial functions as paths

proposition *connected_open_polynomial_connected*:
fixes S :: 'a::euclidean_space set
assumes S: open S connected S
and x \in S y \in S
shows \exists g. polynomial_function g \wedge path_image g \subseteq S \wedge pathstart g = x \wedge pathfinish g = y

theorem *Stone_Weierstrass_polynomial_function_subspace*:
fixes f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space
assumes compact S
and conf: continuous_on S f
and 0 < e
and subspace T f ' S \subseteq T
obtains g **where** polynomial_function g g ' S \subseteq T
 \wedge x. x \in S \implies norm(f x - g x) < e

end

8.5 Radon-Nikodým Derivative

```
theory Radon_Nikodym
imports Bochner_Integration
begin
```

```
definition diff_measure :: 'a measure  $\Rightarrow$  'a measure  $\Rightarrow$  'a measure
```

```
where
```

```
diff_measure M N = measure_of (space M) (sets M) ( $\lambda A. \text{emeasure } M A - \text{emeasure } N A$ )
```

```
proposition (in sigma_finite_measure) obtain_positive_integrable_function:
```

```
obtains f::'a  $\Rightarrow$  real where
```

```
f  $\in$  borel_measurable M
```

```
 $\bigwedge x. f x > 0$ 
```

```
 $\bigwedge x. f x \leq 1$ 
```

```
integrable M f
```

8.5.1 Absolutely continuous

```
definition absolutely_continuous :: 'a measure  $\Rightarrow$  'a measure  $\Rightarrow$  bool where
```

```
absolutely_continuous M N  $\longleftrightarrow$  null_sets M  $\subseteq$  null_sets N
```

8.5.2 Existence of the Radon-Nikodym derivative

```
proposition
```

```
(in finite_measure) Radon_Nikodym_finite_measure:
```

```
assumes finite_measure N and sets_eq[simp]: sets N = sets M
```

```
assumes absolutely_continuous M N
```

```
shows  $\exists f \in \text{borel\_measurable } M. \text{density } M f = N$ 
```

```
proposition (in finite_measure) Radon_Nikodym_finite_measure_infinite:
```

```
assumes absolutely_continuous M N and sets_eq: sets N = sets M
```

```
shows  $\exists f \in \text{borel\_measurable } M. \text{density } M f = N$ 
```

```
theorem (in sigma_finite_measure) Radon_Nikodym:
```

```
assumes ac: absolutely_continuous M N and sets_eq: sets N = sets M
```

```
shows  $\exists f \in \text{borel\_measurable } M. \text{density } M f = N$ 
```

8.5.3 Uniqueness of densities

```
proposition (in sigma_finite_measure) density_unique:
```

```
assumes f: f  $\in$  borel_measurable M
```

```
assumes f': f'  $\in$  borel_measurable M
```

```
assumes density_eq: density M f = density M f'
```

```
shows  $\forall x \text{ in } M. f x = f' x$ 
```

8.5.4 Radon-Nikodym derivative

definition $RN_deriv :: 'a\ measure \Rightarrow 'a\ measure \Rightarrow 'a \Rightarrow ennreal$ **where**

$RN_deriv\ M\ N =$

(if $\exists f. f \in borel_measurable\ M \wedge density\ M\ f = N$

then *SOME* $f. f \in borel_measurable\ M \wedge density\ M\ f = N$

else $(\lambda_. 0)$)

proposition (in *sigma_finite_measure*) *real_RN_deriv*:

assumes *finite_measure* N

assumes *ac*: *absolutely_continuous* $M\ N$ sets $N =$ sets M

obtains D **where** $D \in borel_measurable\ M$

and *AE* x in $M. RN_deriv\ M\ N\ x = ennreal\ (D\ x)$

and *AE* x in $N. 0 < D\ x$

and $\bigwedge x. 0 \leq D\ x$

end

Chapter 9

Integrals over a Set

```
theory Set_Integral
  imports Radon_Nikodym
begin
```

9.1 Notation

```
definition set_borel_measurable M A f ≡ (λx. indicator A x *R f x) ∈ borel_measurable M
```

```
definition set_integrable M A f ≡ integrable M (λx. indicator A x *R f x)
```

```
definition set_lebesgue_integral M A f ≡ lebesgue_integral M (λx. indicator A x *R f x)
```

9.2 Basic properties

```
proposition set_borel_measurable_subset:
  fixes f :: _ ⇒ _ :: {banach, second_countable_topology}
  assumes [measurable]: set_borel_measurable M A f B ∈ sets M and B ⊆ A
  shows set_borel_measurable M B f
```

9.3 Complex integrals

9.4 NN Set Integrals

proposition *nn_integral_disjoint_family*:

assumes *[measurable]*: $f \in \text{borel_measurable } M \wedge (n::\text{nat}). B\ n \in \text{sets } M$
and *disjoint_family* B
shows $(\int^+ x \in (\bigcup n. B\ n). f\ x\ \partial M) = (\sum n. (\int^+ x \in B\ n. f\ x\ \partial M))$

9.5 Scheffé's lemma

proposition *Scheffe_lemma1*:

assumes $\bigwedge n. \text{integrable } M (F\ n) \text{ integrable } M f$
 $AE\ x\ \text{in } M. (\lambda n. F\ n\ x) \longrightarrow f\ x$
 $\text{limsup } (\lambda n. \int^+ x. \text{norm}(F\ n\ x)\ \partial M) \leq (\int^+ x. \text{norm}(f\ x)\ \partial M)$
shows $(\lambda n. \int^+ x. \text{norm}(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

proposition *Scheffe_lemma2*:

fixes $F::\text{nat} \Rightarrow 'a \Rightarrow 'b::\{\text{banach, second_countable_topology}\}$
assumes $\bigwedge n::\text{nat}. F\ n \in \text{borel_measurable } M \text{ integrable } M f$
 $AE\ x\ \text{in } M. (\lambda n. F\ n\ x) \longrightarrow f\ x$
 $\bigwedge n. (\int^+ x. \text{norm}(F\ n\ x)\ \partial M) \leq (\int^+ x. \text{norm}(f\ x)\ \partial M)$
shows $(\lambda n. \int^+ x. \text{norm}(F\ n\ x - f\ x)\ \partial M) \longrightarrow 0$

9.6 Convergence of integrals over an interval

proposition *tendsto_set_lebesgue_integral_at_top*:

fixes $f :: \text{real} \Rightarrow 'a::\{\text{banach, second_countable_topology}\}$
assumes *sets*: $\bigwedge b. b \geq a \implies \{a..b\} \in \text{sets } M$
and *int*: $\text{set_integrable } M \{a..\} f$
shows $((\lambda b. \text{set_lebesgue_integral } M \{a..b\} f) \longrightarrow \text{set_lebesgue_integral } M \{a..\} f) \text{ at_top}$

proposition *tendsto_set_lebesgue_integral_at_bot*:

fixes $f :: \text{real} \Rightarrow 'a::\{\text{banach, second_countable_topology}\}$
assumes *sets*: $\bigwedge a. a \leq b \implies \{a..b\} \in \text{sets } M$
and *int*: $\text{set_integrable } M \{..b\} f$
shows $((\lambda a. \text{set_lebesgue_integral } M \{a..b\} f) \longrightarrow \text{set_lebesgue_integral } M \{..b\} f) \text{ at_bot}$

theorem *integral_Markov_inequality'*:

fixes $u :: 'a \Rightarrow \text{real}$
assumes *[measurable]*: $\text{set_integrable } M\ A\ u$ **and** $A \in \text{sets } M$
assumes $AE\ x\ \text{in } M. x \in A \implies u\ x \geq 0$ **and** $0 < (c::\text{real})$

shows $\text{emeasure } M \{x \in A. u \ x \geq c\} \leq (1/c::\text{real}) * (\int x \in A. u \ x \ \partial M)$

theorem *integral_Markov_inequality'_measure*:

assumes *[measurable]*: $\text{set_integrable } M \ A \ u$ **and** $A \in \text{sets } M$

and $\forall x \in M. x \in A \longrightarrow 0 \leq u \ x \ 0 < (c::\text{real})$

shows $\text{measure } M \{x \in A. u \ x \geq c\} \leq (\int x \in A. u \ x \ \partial M) / c$

theorem (*in finite_measure*) *Chernoff_ineq_ge*:

assumes $s > 0$

assumes *integrable*: $\text{set_integrable } M \ A \ (\lambda x. \exp (s * f \ x))$ **and** $A \in \text{sets } M$

shows $\text{measure } M \{x \in A. f \ x \geq a\} \leq \exp (-s * a) * (\int x \in A. \exp (s * f \ x) \ \partial M)$

proof –

have $\{x \in A. f \ x \geq a\} = \{x \in A. \exp (s * f \ x) \geq \exp (s * a)\}$

using s **by** *auto*

also have $\text{measure } M \dots \leq \text{set_lebesgue_integral } M \ A \ (\lambda x. \exp (s * f \ x)) / \exp (s * a)$

by (*intro integral_Markov_inequality'_measure assms*) *auto*

finally show *?thesis*

by (*simp add: exp_minus_field_simps*)

qed

theorem (*in finite_measure*) *Chernoff_ineq_le*:

assumes $s > 0$

assumes *integrable*: $\text{set_integrable } M \ A \ (\lambda x. \exp (-s * f \ x))$ **and** $A \in \text{sets } M$

shows $\text{measure } M \{x \in A. f \ x \leq a\} \leq \exp (s * a) * (\int x \in A. \exp (-s * f \ x) \ \partial M)$

proof –

have $\{x \in A. f \ x \leq a\} = \{x \in A. \exp (-s * f \ x) \geq \exp (-s * a)\}$

using s **by** *auto*

also have $\text{measure } M \dots \leq \text{set_lebesgue_integral } M \ A \ (\lambda x. \exp (-s * f \ x)) / \exp (-s * a)$

by (*intro integral_Markov_inequality'_measure assms*) *auto*

finally show *?thesis*

by (*simp add: exp_minus_field_simps*)

qed

9.7 Integrable Simple Functions

lemma *integrable_simple_function_induct*[*consumes 2, case_names cong indicator add, induct set: simple_function*]:

fixes $f :: 'a \Rightarrow 'b :: \{\text{second_countable_topology, banach}\}$

assumes f : $\text{simple_function } M \ f \ \text{emeasure } M \ \{y \in \text{space } M. f \ y \neq 0\} \neq \infty$

assumes cong : $\bigwedge f \ g. \text{simple_function } M \ f \Longrightarrow \text{emeasure } M \ \{y \in \text{space } M. f \ y \neq 0\} \neq \infty$

$\Longrightarrow \text{simple_function } M \ g \Longrightarrow \text{emeasure } M \ \{y \in \text{space } M. g \ y \neq$

$0\} \neq \infty$

$\Longrightarrow (\bigwedge x. x \in \text{space } M \Longrightarrow f \ x = g \ x) \Longrightarrow P \ f \Longrightarrow P \ g$

assumes indicator : $\bigwedge A \ y. A \in \text{sets } M \Longrightarrow \text{emeasure } M \ A < \infty \Longrightarrow P \ (\lambda x. \text{indicator } A \ x \ *_R \ y)$

assumes add : $\bigwedge f \ g. \text{simple_function } M \ f \Longrightarrow \text{emeasure } M \ \{y \in \text{space } M. f \ y \neq$

$0\} \neq \infty \implies$
 $\neq \infty \implies$
 $(g z)) \implies$
 $P f \implies P g \implies P (\lambda x. f x + g x)$
shows $P f$
lemma *integrable_simple_function_induct_nn*[consumes 3, case_names *cong indicator add*, induct set: *simple_function*]:
fixes $f :: 'a \Rightarrow 'b :: \{\text{second_countable_topology, banach, linorder_topology, ordered_real_vector}\}$
assumes f : *simple_function* $M f$ *emeasure* $M \{y \in \text{space } M. f y \neq 0\} \neq \infty \wedge x. x \in \text{space } M \longrightarrow f x \geq 0$
assumes *cong*: $\wedge f g. \text{simple_function } M f \implies \text{emeasure } M \{y \in \text{space } M. f y \neq 0\} \neq \infty \implies (\wedge x. x \in \text{space } M \implies f x \geq 0) \implies \text{simple_function } M g \implies \text{emeasure } M \{y \in \text{space } M. g y \neq 0\} \neq \infty \implies (\wedge x. x \in \text{space } M \implies g x \geq 0) \implies (\wedge x. x \in \text{space } M \implies f x = g x) \implies P f \implies P g$
assumes *indicator*: $\wedge A y. y \geq 0 \implies A \in \text{sets } M \implies \text{emeasure } M A < \infty \implies P (\lambda x. \text{indicator } A x *_R y)$
assumes *add*: $\wedge f g. (\wedge x. x \in \text{space } M \implies f x \geq 0) \implies \text{simple_function } M f \implies \text{emeasure } M \{y \in \text{space } M. f y \neq 0\} \neq \infty \implies (\wedge x. x \in \text{space } M \implies g x \geq 0) \implies \text{simple_function } M g \implies \text{emeasure } M \{y \in \text{space } M. g y \neq 0\} \neq \infty \implies (\wedge z. z \in \text{space } M \implies \text{norm } (f z + g z) = \text{norm } (f z) + \text{norm } (g z)) \implies P f \implies P g \implies P (\lambda x. f x + g x)$
shows $P f$

9.7.1 Totally Ordered Banach Spaces

9.7.2 Auxiliary Lemmas for Set Integrals

9.7.3 Integrability and Measurability of the Diameter

9.7.4 Averaging Theorem

corollary *integral_nonneg_eq_0_iff_AE_banach*:

fixes $f :: 'a \Rightarrow 'b :: \{\text{second_countable_topology, banach, linorder_topology, ordered_real_vector}\}$

assumes f [*measurable*]: *integrable* $M f$ **and** *nonneg*: $AE x \text{ in } M. 0 \leq f x$

shows $\text{integral}^L M f = 0 \iff (AE x \text{ in } M. f x = 0)$

corollary *integral_eq_mono_AE_eq_AE*:

fixes $f g :: 'a \Rightarrow 'b :: \{\text{second_countable_topology, banach, linorder_topology, ordered_real_vector}\}$

assumes *integrable* $M f$ *integrable* $M g$ $\text{integral}^L M f = \text{integral}^L M g$ $AE x \text{ in } M. f x \leq g x$

shows $AE x \text{ in } M. f x = g x$

end

9.8 Homeomorphism Theorems

theory *Homeomorphism*
imports *Homotopy*
begin

9.8.1 Homeomorphism of all convex compact sets with nonempty interior

proposition

fixes $S :: 'a::euclidean_space\ set$
assumes *compact* S **and** $0: 0 \in rel_interior\ S$
and $star: \bigwedge x. x \in S \implies open_segment\ 0\ x \subseteq rel_interior\ S$
shows *starlike_compact_projective1_0*:
 $S - rel_interior\ S$ *homeomorphic sphere* $0\ 1 \cap affine\ hull\ S$
(is ?SMINUS homeomorphic ?SPHER)
and *starlike_compact_projective2_0*:
 S *homeomorphic cball* $0\ 1 \cap affine\ hull\ S$
(is S homeomorphic ?CBALL)

corollary

fixes $S :: 'a::euclidean_space\ set$
assumes *compact* S **and** $a: a \in rel_interior\ S$
and $star: \bigwedge x. x \in S \implies open_segment\ a\ x \subseteq rel_interior\ S$
shows *starlike_compact_projective1*:
 $S - rel_interior\ S$ *homeomorphic sphere* $a\ 1 \cap affine\ hull\ S$
and *starlike_compact_projective2*:
 S *homeomorphic cball* $a\ 1 \cap affine\ hull\ S$

corollary *starlike_compact_projective_special*:

assumes *compact* S
and $cb01: cball\ (0::'a::euclidean_space)\ 1 \subseteq S$
and $scale: \bigwedge x\ u. \llbracket x \in S; 0 \leq u; u < 1 \rrbracket \implies u *_R\ x \in S - frontier\ S$
shows S *homeomorphic* $(cball\ (0::'a::euclidean_space)\ 1)$

9.8.2 Homeomorphisms between punctured spheres and affine sets

theorem *homeomorphic_punctured_affine_sphere_affine*:

fixes $a :: 'a :: euclidean_space$
assumes $0 < r\ b \in sphere\ a\ r\ affine\ T\ a \in T\ b \in T\ affine\ p$
and $aff: aff_dim\ T = aff_dim\ p + 1$
shows $(sphere\ a\ r \cap T) - \{b\}$ *homeomorphic* p

corollary *homeomorphic_punctured_sphere_affine:*

fixes $a :: 'a :: \text{euclidean_space}$
assumes $0 < r$ **and** $b \in \text{sphere } a \ r$
and *affine* T **and** *affS*: $\text{aff_dim } T + 1 = \text{DIM}('a)$
shows $(\text{sphere } a \ r - \{b\})$ *homeomorphic* T

corollary *homeomorphic_punctured_sphere_hyperplane:*

fixes $a :: 'a :: \text{euclidean_space}$
assumes $0 < r$ **and** $b \in \text{sphere } a \ r$
and $c \neq 0$
shows $(\text{sphere } a \ r - \{b\})$ *homeomorphic* $\{x :: 'a. c \cdot x = d\}$

proposition *homeomorphic_punctured_sphere_affine_gen:*

fixes $a :: 'a :: \text{euclidean_space}$
assumes *convex* S *bounded* S **and** $a: a \in \text{rel_frontier } S$
and *affine* T **and** *affS*: $\text{aff_dim } S = \text{aff_dim } T + 1$
shows $\text{rel_frontier } S - \{a\}$ *homeomorphic* T

proposition *homeomorphic_closedin_convex:*

fixes $S :: 'm :: \text{euclidean_space set}$
assumes $\text{aff_dim } S < \text{DIM}('n)$
obtains U **and** $T :: 'n :: \text{euclidean_space set}$
where *convex* U $U \neq \{\}$ *closedin* $(\text{top_of_set } U)$ T
 S *homeomorphic* T

9.8.3 Locally compact sets in an open set

proposition *locally_compact_homeomorphic_closed:*

fixes $S :: 'a :: \text{euclidean_space set}$
assumes *locally compact* S **and** *dimlt*: $\text{DIM}('a) < \text{DIM}('b)$
obtains $T :: 'b :: \text{euclidean_space set}$ **where** *closed* T S *homeomorphic* T

proposition *homeomorphic_convex_compact_cball:*

fixes $e :: \text{real}$
and $S :: 'a :: \text{euclidean_space set}$
assumes S : *convex* S *compact* S *interior* $S \neq \{\}$ **and** $e > 0$
shows S *homeomorphic* $(\text{cball } (b :: 'a) \ e)$

corollary *homeomorphic_convex_compact:*

fixes $S :: 'a :: \text{euclidean_space set}$
and $T :: 'a \text{ set}$
assumes *convex* S *compact* S *interior* $S \neq \{\}$
and *convex* T *compact* T *interior* $T \neq \{\}$
shows S *homeomorphic* T

9.8.4 Covering spaces and lifting results for them

definition *covering_space*

$:: 'a::\text{topological_space set} \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b::\text{topological_space set} \Rightarrow \text{bool}$

where

$\text{covering_space } c \ p \ S \equiv$

$\text{continuous_on } c \ p \wedge p \text{ ' } c = S \wedge$
 $(\forall x \in S. \exists T. x \in T \wedge \text{openin } (\text{top_of_set } S) \ T \wedge$
 $(\exists v. \bigcup v = c \cap p \text{ ' } T \wedge$
 $(\forall u \in v. \text{openin } (\text{top_of_set } c) \ u) \wedge$
 $\text{pairwise disjoint } v \wedge$
 $(\forall u \in v. \exists q. \text{homeomorphism } u \ T \ p \ q)))$

proposition *covering_space_open_map*:

fixes $S :: 'a :: \text{metric_space set}$ **and** $T :: 'b :: \text{metric_space set}$

assumes $p: \text{covering_space } c \ p \ S$ **and** $T: \text{openin } (\text{top_of_set } c) \ T$

shows $\text{openin } (\text{top_of_set } S) \ (p \text{ ' } T)$

proposition *covering_space_lift_unique*:

fixes $f :: 'a::\text{topological_space} \Rightarrow 'b::\text{topological_space}$

fixes $g1 :: 'a \Rightarrow 'c::\text{real_normed_vector}$

assumes $\text{covering_space } c \ p \ S$

$g1 \ a = g2 \ a$

$\text{continuous_on } T \ f \ f \in T \rightarrow S$

$\text{continuous_on } T \ g1 \ g1 \in T \rightarrow c \ \wedge x. x \in T \Longrightarrow f \ x = p(g1 \ x)$

$\text{continuous_on } T \ g2 \ g2 \in T \rightarrow c \ \wedge x. x \in T \Longrightarrow f \ x = p(g2 \ x)$

$\text{connected } T \ a \in T \ x \in T$

shows $g1 \ x = g2 \ x$

proposition *covering_space_locally_eq*:

fixes $p :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{real_normed_vector}$

assumes $\text{cov}: \text{covering_space } C \ p \ S$

and $\text{pim}: \bigwedge T. \llbracket T \subseteq C; \varphi \ T \rrbracket \Longrightarrow \psi(p \text{ ' } T)$

and $\text{qim}: \bigwedge q \ U. \llbracket U \subseteq S; \text{continuous_on } U \ q; \psi \ U \rrbracket \Longrightarrow \varphi(q \text{ ' } U)$

shows $\text{locally } \psi \ S \longleftrightarrow \text{locally } \varphi \ C$

(**is** $?lhs = ?rhs$)

proposition *covering_space_lift_homotopy*:

fixes $p :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{real_normed_vector}$

and $h :: \text{real} \times 'c::\text{real_normed_vector} \Rightarrow 'b$

assumes $\text{cov}: \text{covering_space } C \ p \ S$

and $\text{conth}: \text{continuous_on } (\{0..1\} \times U) \ h$

and $\text{him}: h \in (\{0..1\} \times U) \rightarrow S$

and *heq*: $\bigwedge y. y \in U \implies h(0, y) = p(f y)$
and *contf*: *continuous_on* $U f$ **and** *fim*: $f \in U \rightarrow C$
obtains *k* **where** *continuous_on* $(\{0..1\} \times U) k$
 $k \in (\{0..1\} \times U) \rightarrow C$
 $\bigwedge y. y \in U \implies k(0, y) = f y$
 $\bigwedge z. z \in \{0..1\} \times U \implies h z = p(k z)$

corollary *covering_space_lift_homotopy_alt*:

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
and $h :: 'c::real_normed_vector \times real \Rightarrow 'b$
assumes *cov*: *covering_space* $C p S$
and *conth*: *continuous_on* $(U \times \{0..1\}) h$
and *him*: $h \in (U \times \{0..1\}) \rightarrow S$
and *heq*: $\bigwedge y. y \in U \implies h(y, 0) = p(f y)$
and *contf*: *continuous_on* $U f$ **and** *fim*: $f \in U \rightarrow C$
obtains *k* **where** *continuous_on* $(U \times \{0..1\}) k$
 $k \in (U \times \{0..1\}) \rightarrow C$
 $\bigwedge y. y \in U \implies k(y, 0) = f y$
 $\bigwedge z. z \in U \times \{0..1\} \implies h z = p(k z)$

corollary *covering_space_lift_homotopic_function*:

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$ **and** $g :: 'c::real_normed_vector \Rightarrow 'a$
assumes *cov*: *covering_space* $C p S$
and *contg*: *continuous_on* $U g$
and *gim*: $g \in U \rightarrow C$
and *pgeq*: $\bigwedge y. y \in U \implies p(g y) = f y$
and *hom*: *homotopic_with_canon* $(\lambda x. True) U S f f'$
obtains *g'* **where** *continuous_on* $U g'$ *image* $g' U \subseteq C$ $\bigwedge y. y \in U \implies p(g' y) = f' y$

corollary *covering_space_lift_inessential_function*:

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$ **and** $U :: 'c::real_normed_vector$ *set*
assumes *cov*: *covering_space* $C p S$
and *hom*: *homotopic_with_canon* $(\lambda x. True) U S f (\lambda x. a)$
obtains *g* **where** *continuous_on* $U g$ $g' U \subseteq C$ $\bigwedge y. y \in U \implies p(g y) = f y$

9.8.5 Lifting of general functions to covering space

proposition *covering_space_lift_path_strong*:

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
and $f :: 'c::real_normed_vector \Rightarrow 'b$
assumes *cov*: *covering_space* $C p S$ **and** $a \in C$
and *path g* **and** *pag*: *path_image* $g \subseteq S$ **and** *pas*: *pathstart* $g = p a$
obtains *h* **where** *path h* *path_image* $h \subseteq C$ *pathstart* $h = a$
and $\bigwedge t. t \in \{0..1\} \implies p(h t) = g t$

corollary *covering_space_lift_path:*

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes $cov: covering_space\ C\ p\ S$ **and** $path\ g$ **and** $pig: path_image\ g \subseteq S$
obtains h **where** $path\ h\ path_image\ h \subseteq C \wedge t. t \in \{0..1\} \Longrightarrow p(h\ t) = g\ t$

proposition *covering_space_lift_homotopic_paths:*

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes $cov: covering_space\ C\ p\ S$
and $path\ g1$ **and** $pig1: path_image\ g1 \subseteq S$
and $path\ g2$ **and** $pig2: path_image\ g2 \subseteq S$
and $hom: homotopic_paths\ S\ g1\ g2$
and $path\ h1$ **and** $pih1: path_image\ h1 \subseteq C$ **and** $ph1: \wedge t. t \in \{0..1\} \Longrightarrow$
 $p(h1\ t) = g1\ t$
and $path\ h2$ **and** $pih2: path_image\ h2 \subseteq C$ **and** $ph2: \wedge t. t \in \{0..1\} \Longrightarrow$
 $p(h2\ t) = g2\ t$
and $h1h2: pathstart\ h1 = pathstart\ h2$
shows $homotopic_paths\ C\ h1\ h2$

corollary *covering_space_monodromy:*

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes $cov: covering_space\ C\ p\ S$
and $path\ g1$ **and** $pig1: path_image\ g1 \subseteq S$
and $path\ g2$ **and** $pig2: path_image\ g2 \subseteq S$
and $hom: homotopic_paths\ S\ g1\ g2$
and $path\ h1$ **and** $pih1: path_image\ h1 \subseteq C$ **and** $ph1: \wedge t. t \in \{0..1\} \Longrightarrow$
 $p(h1\ t) = g1\ t$
and $path\ h2$ **and** $pih2: path_image\ h2 \subseteq C$ **and** $ph2: \wedge t. t \in \{0..1\} \Longrightarrow$
 $p(h2\ t) = g2\ t$
and $h1h2: pathstart\ h1 = pathstart\ h2$
shows $pathfinish\ h1 = pathfinish\ h2$

corollary *covering_space_lift_homotopic_path:*

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
assumes $cov: covering_space\ C\ p\ S$
and $hom: homotopic_paths\ S\ f\ f'$
and $path\ g$ **and** $pig: path_image\ g \subseteq C$
and $a: pathstart\ g = a$ **and** $b: pathfinish\ g = b$
and $pgeq: \wedge t. t \in \{0..1\} \Longrightarrow p(g\ t) = f\ t$
obtains g' **where** $path\ g'\ path_image\ g' \subseteq C$
 $pathstart\ g' = a\ pathfinish\ g' = b \wedge t. t \in \{0..1\} \Longrightarrow p(g'\ t) = f'\ t$

proposition *covering_space_lift_general:*

fixes $p :: 'a::real_normed_vector \Rightarrow 'b::real_normed_vector$
and $f :: 'c::real_normed_vector \Rightarrow 'b$
assumes $cov: covering_space\ C\ p\ S$ **and** $a \in C\ z \in U$

and U : *path_connected* U *locally path_connected* U
and *contf*: *continuous_on* U f **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f z = p a$
and *hom*: $\bigwedge r. \llbracket \text{path } r; \text{path_image } r \subseteq U; \text{pathstart } r = z; \text{pathfinish } r = z \rrbracket$
 $\implies \exists q. \text{path } q \wedge \text{path_image } q \subseteq C \wedge$
 $\text{pathstart } q = a \wedge \text{pathfinish } q = a \wedge$
 $\text{homotopic_paths } S (f \circ r) (p \circ q)$
obtains g **where** *continuous_on* U g $g \in U \rightarrow C$ $g z = a$ $\bigwedge y. y \in U \implies p(g$
 $y) = f y$

corollary *covering_space_lift_stronger*:

fixes $p :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{real_normed_vector}$
and $f :: 'c::\text{real_normed_vector} \Rightarrow 'b$
assumes *cov*: *covering_space* C p S $a \in C$ $z \in U$
and U : *path_connected* U *locally path_connected* U
and *contf*: *continuous_on* U f **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f z = p a$
and *hom*: $\bigwedge r. \llbracket \text{path } r; \text{path_image } r \subseteq U; \text{pathstart } r = z; \text{pathfinish } r = z \rrbracket$
 $\implies \exists b. \text{homotopic_paths } S (f \circ r) (\text{linepath } b b)$
obtains g **where** *continuous_on* U g $g \in U \rightarrow C$ $g z = a$ $\bigwedge y. y \in U \implies p(g$
 $y) = f y$

corollary *covering_space_lift_strong*:

fixes $p :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{real_normed_vector}$
and $f :: 'c::\text{real_normed_vector} \Rightarrow 'b$
assumes *cov*: *covering_space* C p S $a \in C$ $z \in U$
and *scU*: *simply_connected* U **and** *lpcU*: *locally path_connected* U
and *contf*: *continuous_on* U f **and** *fim*: $f \in U \rightarrow S$
and *feq*: $f z = p a$
obtains g **where** *continuous_on* U g $g \in U \rightarrow C$ $g z = a$ $\bigwedge y. y \in U \implies p(g$
 $y) = f y$

corollary *covering_space_lift*:

fixes $p :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{real_normed_vector}$
and $f :: 'c::\text{real_normed_vector} \Rightarrow 'b$
assumes *cov*: *covering_space* C p S
and U : *simply_connected* U *locally path_connected* U
and *contf*: *continuous_on* U f **and** *fim*: $f \in U \rightarrow S$
obtains g **where** *continuous_on* U g $g \in U \rightarrow C$ $\bigwedge y. y \in U \implies p(g y) = f y$

end

theory *Equivalence_Lebesgue_Henstock_Integration*

imports

Lebesgue_Measure

Henstock_Kurzweil_Integration

Complete_Measure

Set_Integral

Homeomorphism
Cartesian_Euclidean_Space
begin

9.8.6 Equivalence Lebesgue integral on *lborel* and HK-integral

9.8.7 Absolute integrability (this is the same as Lebesgue integrability)

9.8.8 Applications to Negligibility

corollary *eventually_ae_filter_negligible*:

eventually P (ae_filter lebesgue) \longleftrightarrow ($\exists N$. negligible $N \wedge \{x. \neg P x\} \subseteq N$)

proposition *negligible_convex_frontier*:

fixes $S :: 'N :: euclidean_space$ set

assumes *convex S*

shows *negligible(frontier S)*

corollary *negligible_sphere*: *negligible (sphere a e)*

proposition *open_not_negligible*:

assumes *open S S $\neq \{\}$*

shows \neg *negligible S*

9.8.9 Negligibility of image under non-injective linear map

9.8.10 Negligibility of a Lipschitz image of a negligible set

proposition *negligible_locally_Lipschitz_image*:

fixes $f :: 'M :: euclidean_space \Rightarrow 'N :: euclidean_space$

assumes $M \leq N$: $DIM('M) \leq DIM('N)$ *negligible S*

and *lips: $\bigwedge x. x \in S$*

$\implies \exists T B$. *open T $\wedge x \in T \wedge$*

$(\forall y \in S \cap T. norm(f y - f x) \leq B * norm(y - x))$

shows *negligible (f ' S)*

corollary *negligible_differentiable_image_negligible*:

fixes $f :: 'M :: euclidean_space \Rightarrow 'N :: euclidean_space$

assumes $M \leq N$: $DIM('M) \leq DIM('N)$ *negligible S*

and *diff_f: f differentiable_on S*

shows *negligible (f ' S)*

corollary *negligible_differentiable_image_lowdim:*
fixes $f :: 'M::\text{euclidean_space} \Rightarrow 'N::\text{euclidean_space}$
assumes $M \text{ less } N: \text{DIM}('M) < \text{DIM}('N)$ **and** $\text{diff_}f: f \text{ differentiable_on } S$
shows $\text{negligible } (f \text{ ' } S)$

9.8.11 Measurability of countable unions and intersections of various kinds.

9.8.12 Negligibility is a local property

9.8.13 Integral bounds

proposition *bounded_variation_absolutely_integrable_interval:*
fixes $f :: 'n::\text{euclidean_space} \Rightarrow 'm::\text{euclidean_space}$
assumes $f: f \text{ integrable_on } \text{cbox } a \text{ } b$
and $*$: $\bigwedge d. d \text{ division_of } (\text{cbox } a \text{ } b) \implies \text{sum } (\lambda K. \text{norm}(\text{integral } K \text{ } f)) d \leq B$
shows $f \text{ absolutely_integrable_on } \text{cbox } a \text{ } b$

9.8.14 Outer and inner approximation of measurable sets by well-behaved sets.

proposition *measurable_outer_intervals_bounded:*
assumes $S \in \text{lmeasurable } S \subseteq \text{cbox } a \text{ } b \text{ } e > 0$
obtains \mathcal{D}
where *countable* \mathcal{D}
 $\bigwedge K. K \in \mathcal{D} \implies K \subseteq \text{cbox } a \text{ } b \wedge K \neq \{\}$ $\wedge (\exists c \text{ } d. K = \text{cbox } c \text{ } d)$
pairwise $(\lambda A \text{ } B. \text{interior } A \cap \text{interior } B = \{\}) \mathcal{D}$
 $\bigwedge u \text{ } v. \text{cbox } u \text{ } v \in \mathcal{D} \implies \exists n. \forall i \in \text{Basis}. v \cdot i - u \cdot i = (b \cdot i - a \cdot i) / 2^n$
 $\bigwedge K. [\![K \in \mathcal{D}; \text{box } a \text{ } b \neq \{\}]\!] \implies \text{interior } K \neq \{\}$
 $S \subseteq \bigcup \mathcal{D} \cup \mathcal{D} \in \text{lmeasurable measure lebesgue } (\bigcup \mathcal{D}) \leq \text{measure lebesgue } S$
 $+ e$

9.8.15 Transformation of measure by linear maps

proposition *measure_linear_sufficient:*
fixes $f :: 'n::\text{euclidean_space} \Rightarrow 'n$
assumes *linear* f **and** $S: S \in \text{lmeasurable}$
and $\text{im}: \bigwedge a \text{ } b. \text{measure lebesgue } (f \text{ ' } (\text{cbox } a \text{ } b)) = m * \text{measure lebesgue } (\text{cbox } a \text{ } b)$
shows $f \text{ ' } S \in \text{lmeasurable} \wedge m * \text{measure lebesgue } S = \text{measure lebesgue } (f \text{ ' } S)$

9.8.16 Lemmas about absolute integrability

corollary *absolutely_integrable_on_const* [simp]:
fixes $c :: 'a::euclidean_space$
assumes $S \in lmeasurable$
shows $(\lambda x. c)$ *absolutely_integrable_on* S

9.8.17 Componentwise

proposition *absolutely_integrable_componentwise_iff*:
shows f *absolutely_integrable_on* $A \longleftrightarrow (\forall b \in Basis. (\lambda x. f\ x \cdot b)$ *absolutely_integrable_on* $A)$

corollary *absolutely_integrable_max_1*:
fixes $f :: 'n::euclidean_space \Rightarrow real$
assumes f *absolutely_integrable_on* S g *absolutely_integrable_on* S
shows $(\lambda x. \max (f\ x) (g\ x))$ *absolutely_integrable_on* S

corollary *absolutely_integrable_min_1*:
fixes $f :: 'n::euclidean_space \Rightarrow real$
assumes f *absolutely_integrable_on* S g *absolutely_integrable_on* S
shows $(\lambda x. \min (f\ x) (g\ x))$ *absolutely_integrable_on* S

9.8.18 Dominated convergence

proposition *integral_countable_UN*:
fixes $f :: real^m \Rightarrow real^n$
assumes f : f *absolutely_integrable_on* $(\bigcup (\text{range } s))$
and s : $\bigwedge m. s\ m \in \text{sets lebesgue}$
shows $\bigwedge n. f$ *absolutely_integrable_on* $(\bigcup_{m \leq n} s\ m)$
and $(\lambda n. \text{integral } (\bigcup_{m \leq n} s\ m) f) \longrightarrow \text{integral } (\bigcup (s \text{ ' } UNIV)) f$ (**is** ? $F \longrightarrow ?I$)

9.8.19 Fundamental Theorem of Calculus for the Lebesgue integral

9.8.20 Integration by parts

9.8.21 A non-negative continuous function whose integral is zero must be zero

corollary *integral_cbox_eq_0_iff:*

fixes $f :: 'a::euclidean_space \Rightarrow real$

assumes *continuous_on* (cbox a b) f **and** $box\ a\ b \neq \{\}$

and $\bigwedge x. x \in cbox\ a\ b \implies f\ x \geq 0$

shows $integral\ (cbox\ a\ b)\ f = 0 \iff (\forall x \in cbox\ a\ b. f\ x = 0)$ (is ?lhs = ?rhs)

9.8.22 Various common equivalent forms of function measurability

9.8.23 Lebesgue sets and continuous images

proposition *lebesgue_regular_inner:*

assumes $S \in sets\ lebesgue$

obtains $K\ C$ **where** *negligible* K $\bigwedge n::nat. compact\ (C\ n)\ S = (\bigcup n. C\ n) \cup K$

9.8.24 Affine lemmas

lemma *lebesgue_integral_real_affine:*

fixes $f :: real \Rightarrow 'a::euclidean_space$ **and** $c :: real$

assumes $c: c \neq 0$ **shows** $(\int x. f\ x\ \partial\ lebesgue) = |c| *_R (\int x. f(t + c * x)\ \partial\ lebesgue)$

9.8.25 More results on integrability

proposition *measurable_bounded_by_integrable_imp_integrable:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$

assumes $f: f \in borel_measurable\ (lebesgue_on\ S)$ **and** $g: g\ integrable_on\ S$

and $normf: \bigwedge x. x \in S \implies norm(f\ x) \leq g\ x$ **and** $S: S \in sets\ lebesgue$

shows $f\ integrable_on\ S$

9.8.26 Relation between Borel measurability and integrability.

proposition *negligible_differentiable_vimage*:
fixes $f :: 'a \Rightarrow 'a::\text{euclidean_space}$
assumes *negligible T*
and $f': \bigwedge x. x \in S \implies \text{inj}(f' x)$
and $\text{derf}: \bigwedge x. x \in S \implies (f \text{ has_derivative } f' x) \text{ (at } x \text{ within } S)$
shows *negligible* $\{x \in S. f x \in T\}$

proposition *has_derivative_inverse_within*:
fixes $f :: 'a::\text{real_normed_vector} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{der_f}: (f \text{ has_derivative } f') \text{ (at } a \text{ within } S)$
and $\text{cont_g}: \text{continuous (at } (f a) \text{ within } f^{-1} S) g$
and $a \in S$ **linear** g' **and** $\text{id}: g' \circ f' = \text{id}$
and $\text{gf}: \bigwedge x. x \in S \implies g(f x) = x$
shows $(g \text{ has_derivative } g') \text{ (at } (f a) \text{ within } f^{-1} S)$

end

9.9 Harmonic Numbers

theory *Harmonic_Numbers*

imports

Complex_Transcendental

Summation_Tests

begin

9.9.1 The Harmonic numbers

definition $\text{harm} :: \text{nat} \Rightarrow 'a :: \text{real_normed_field}$ **where**
 $\text{harm } n = (\sum_{k=1..n. \text{inverse (of_nat } k)})$

theorem *not_convergent_harm*: $\neg \text{convergent (harm :: nat} \Rightarrow 'a :: \text{real_normed_field)}$

9.9.2 The Euler-Mascheroni constant

lemma *euler_mascheroni_LIMSEQ*:
 $(\lambda n. \text{harm } n - \ln (\text{of_nat } n) :: \text{real}) \longrightarrow \text{euler_mascheroni}$

theorem *alternating_harmonic_series_sums*: $(\lambda k. (-1)^k / \text{real_of_nat (Suc } k)) \text{ sums } \ln 2$

end

9.10 The Gamma Function

```

theory Gamma_Function
  imports
    Equivalence_Lebesgue_Henstock_Integration
    Summation_Tests
    Harmonic_Numbers
    HOL-Library.Nonpos_Ints
    HOL-Library.Periodic_Fun
begin

```

9.10.1 The Euler form and the logarithmic Gamma function

definition *Gamma_series* :: ('a :: {banach,real_normed_field}) \Rightarrow nat \Rightarrow 'a **where**
Gamma_series z n = fact n * exp (z * of_real (ln (of_nat n))) / pochhammer z (n+1)

definition *ln_Gamma_series* :: ('a :: {banach,real_normed_field,ln}) \Rightarrow nat \Rightarrow 'a **where**
ln_Gamma_series z n = z * ln (of_nat n) - ln z - ($\sum_{k=1..n}$ ln (z / of_nat (k + 1)))

theorem *ln_Gamma_complex_LIMSEQ*: (z :: complex) $\notin \mathbf{Z}_{\leq 0} \implies \text{ln_Gamma_series } z \longrightarrow \text{ln_Gamma } z$

9.10.2 The Polygamma functions

definition *Polygamma* :: nat \Rightarrow ('a :: {real_normed_field,banach}) \Rightarrow 'a **where**
Polygamma n z = (if n = 0 then
 $(\sum k. \text{inverse } (\text{of_nat } (\text{Suc } k)) - \text{inverse } (z + \text{of_nat } k)) - \text{euler_mascheroni}$
else
 $(-1)^{\wedge \text{Suc } n} * \text{fact } n * (\sum k. \text{inverse } ((z + \text{of_nat } k)^{\wedge \text{Suc } n}))$)

abbreviation *Digamma* :: ('a :: {real_normed_field,banach}) \Rightarrow 'a **where**
Digamma \equiv *Polygamma* 0

theorem *Digamma_LIMSEQ*:
fixes z :: 'a :: {banach,real_normed_field}
assumes z: z \neq 0
shows ($\lambda m. \text{of_real } (\text{ln } (\text{real } m)) - (\sum_{n < m} \text{inverse } (z + \text{of_nat } n))$) \longrightarrow *Digamma* z

theorem *Polygamma_LIMSEQ*:
fixes z :: 'a :: {banach,real_normed_field}
assumes z \neq 0 and n > 0

shows $(\lambda k. \text{inverse } ((z + \text{of_nat } k)^{\wedge \text{Suc } n})) \text{ sums } ((-1)^{\wedge \text{Suc } n} * \text{Polygamma } n \ z / \text{fact } n)$

theorem *has_field_derivative_ln_Gamma_complex* [*derivative_intros*]:
fixes $z :: \text{complex}$
assumes $z: z \notin \mathbb{R}_{\leq 0}$
shows $(\text{ln_Gamma } \text{has_field_derivative } \text{Digamma } z) \text{ (at } z)$

theorem *Polygamma_plus1*:
assumes $z \neq 0$
shows $\text{Polygamma } n \ (z + 1) = \text{Polygamma } n \ z + (-1)^{\wedge n} * \text{fact } n / (z^{\wedge \text{Suc } n})$

theorem *Digamma_of_nat*:
 $\text{Digamma } (\text{of_nat } (\text{Suc } n)) :: 'a :: \{\text{real_normed_field, banach}\} = \text{harm } n - \text{euler_mascheroni}$

theorem *has_field_derivative_Polygamma* [*derivative_intros*]:
fixes $z :: 'a :: \{\text{real_normed_field, euclidean_space}\}$
assumes $z: z \notin \mathbb{Z}_{\leq 0}$
shows $(\text{Polygamma } n \ \text{has_field_derivative } \text{Polygamma } (\text{Suc } n) \ z) \text{ (at } z \text{ within } A)$

9.10.3 Basic properties

theorem *Gamma_series_LIMSEQ* [*tendsto_intros*]:
 $\text{Gamma_series } z \longrightarrow \text{Gamma } z$

theorem *Gamma_plus1*: $z \notin \mathbb{Z}_{\leq 0} \implies \text{Gamma } (z + 1) = z * \text{Gamma } z$

theorem *pochhammer_Gamma*: $z \notin \mathbb{Z}_{\leq 0} \implies \text{pochhammer } z \ n = \text{Gamma } (z + \text{of_nat } n) / \text{Gamma } z$

theorem *Gamma_fact*: $\text{Gamma } (1 + \text{of_nat } n) = \text{fact } n$

9.10.4 Differentiability

theorem *has_field_derivative_Gamma* [*derivative_intros*]:
 $z \notin \mathbb{Z}_{\leq 0} \implies (\text{Gamma } \text{has_field_derivative } \text{Gamma } z * \text{Digamma } z) \text{ (at } z \text{ within } A)$

theorem *log_convex_Gamma_real*: *convex_on* {0<..} (*ln* ∘ *Gamma* :: *real* ⇒ *real*)

9.10.5 The uniqueness of the real Gamma function

theorem *Gamma_pos_real_unique*:
assumes $x > 0$
shows $G\ x = \text{Gamma}\ x$

9.10.6 The Beta function

theorem *Beta_plus1_plus1*:
assumes $x \notin \mathbb{Z}_{\leq 0}$ $y \notin \mathbb{Z}_{\leq 0}$
shows $\text{Beta}\ (x + 1)\ y + \text{Beta}\ x\ (y + 1) = \text{Beta}\ x\ y$

theorem *Beta_plus1_left*:
assumes $x \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * \text{Beta}\ (x + 1)\ y = x * \text{Beta}\ x\ y$

theorem *Beta_plus1_right*:
assumes $y \notin \mathbb{Z}_{\leq 0}$
shows $(x + y) * \text{Beta}\ x\ (y + 1) = y * \text{Beta}\ x\ y$

9.10.7 Legendre duplication theorem

theorem *Gamma_legendre_duplication*:
fixes $z :: \text{complex}$
assumes $z \notin \mathbb{Z}_{\leq 0}$ $z + 1/2 \notin \mathbb{Z}_{\leq 0}$
shows $\text{Gamma}\ z * \text{Gamma}\ (z + 1/2) =$
 $\exp\ ((1 - 2*z) * \text{of_real}\ (\ln\ 2)) * \text{of_real}\ (\text{sqrt}\ \pi) * \text{Gamma}\ (2*z)$

9.10.8 Alternative definitions

theorem *Gamma_series_euler'*:
assumes $z: (z :: 'a :: \text{Gamma}) \notin \mathbb{Z}_{\leq 0}$
shows $(\lambda n. \text{Gamma_series_euler}'\ z\ n) \longrightarrow \text{Gamma}\ z$

theorem *Gamma_Weierstrass_complex*: *Gamma_series_Weierstrass* $z \longrightarrow$
Gamma ($z :: \text{complex}$)

theorem *gbinomial_Gamma*:

assumes $z + 1 \notin \mathbb{Z}_{\leq 0}$

shows $(z \text{ gchoose } n) = \text{Gamma } (z + 1) / (\text{fact } n * \text{Gamma } (z - \text{of_nat } n + 1))$

theorem *Gamma_integral_complex*:

assumes $z: \text{Re } z > 0$

shows $((\lambda t. \text{of_real } t \text{ powr } (z - 1) / \text{of_real } (\text{exp } t)) \text{ has_integral } \text{Gamma } z)$
 $\{0..\}$

theorem *has_integral_Beta_real*:

assumes $a: a > 0$ **and** $b: b > (0 :: \text{real})$

shows $((\lambda t. t \text{ powr } (a - 1) * (1 - t) \text{ powr } (b - 1)) \text{ has_integral } \text{Beta } a \ b)$
 $\{0..1\}$

9.10.9 The Weierstraß product formula for the sine

theorem *sin_product_formula_complex*:

fixes $z :: \text{complex}$

shows $(\lambda n. \text{of_real } \pi * z * (\prod_{k=1..n}. (1 - z^2 / \text{of_nat } k^2))) \longrightarrow \text{sin}$
 $(\text{of_real } \pi * z)$

theorem *wallis*: $(\lambda n. \prod_{k=1..n}. (4 * \text{real } k^2) / (4 * \text{real } k^2 - 1)) \longrightarrow \pi / 2$

9.10.10 The Solution to the Basel problem

theorem *inverse_squares_sums*: $(\lambda n. 1 / (n + 1)^2) \text{ sums } (\pi^2 / 6)$

end

theory *Interval_Integral*

imports *Equivalence_Lebesgue_Henstock_Integration*

begin

9.10.11 Approximating a (possibly infinite) interval

proposition *einterval_Icc_approximation*:

fixes $a \ b :: \text{ereal}$

assumes $a < b$

obtains $u \ l :: \text{nat} \Rightarrow \text{real}$ **where**

$einterval \ a \ b = (\bigcup i. \{l \ i .. u \ i\})$

$incseq \ u \ decseq \ l \ \wedge i. l \ i < u \ i \ \wedge i. a < l \ i \ \wedge i. u \ i < b$

$$l \longrightarrow a \quad u \longrightarrow b$$

definition *interval_lebesgue_integral* :: *real measure* \Rightarrow *ereal* \Rightarrow *ereal* \Rightarrow (*real* \Rightarrow 'a) \Rightarrow 'a::{*banach, second_countable_topology*} **where**
interval_lebesgue_integral M a b f =
 (if $a \leq b$ then (LINT x:einterval a b|M. f x) else - (LINT x:einterval b a|M. f x))

definition *interval_lebesgue_integrable* :: *real measure* \Rightarrow *ereal* \Rightarrow *ereal* \Rightarrow (*real* \Rightarrow 'a::{*banach, second_countable_topology*}) \Rightarrow *bool* **where**
interval_lebesgue_integrable M a b f =
 (if $a \leq b$ then *set_integrable* M (einterval a b) f else *set_integrable* M (einterval b a) f)

9.10.12 Basic properties of integration over an interval

proposition *interval_integrable_to_infinity_eq*: (*interval_lebesgue_integrable* M a ∞ f) =
 (*set_integrable* M {a<..} f)

9.10.13 Basic properties of integration over an interval wrt lebesgue measure

9.10.14 General limit approximation arguments

proposition *interval_integral_Icc_approx_nonneg*:
fixes a b :: *ereal*
assumes a < b
fixes u l :: *nat* \Rightarrow *real*
assumes approx: einterval a b = (\bigcup i. {l i .. u i})
 incseq u decseq l \wedge i. l i < u i \wedge i. a < l i \wedge i. u i < b
 l \longrightarrow a u \longrightarrow b
fixes f :: *real* \Rightarrow *real*
assumes f_integrable: \bigwedge i. *set_integrable* lborel {l i .. u i} f
assumes f_nonneg: AE x in lborel. a < *ereal* x \longrightarrow *ereal* x < b \longrightarrow 0 \leq f x
assumes f_measurable: *set_borel_measurable* lborel (einterval a b) f
assumes lbint_lim: (λ i. LBINT x=l i .. u i. f x) \longrightarrow C
shows
set_integrable lborel (einterval a b) f
 (LBINT x=a..b. f x) = C

proposition *interval_integral_Icc_approx_integrable*:

fixes $u\ l :: \text{nat} \Rightarrow \text{real}$ **and** $a\ b :: \text{ereal}$
fixes $f :: \text{real} \Rightarrow 'a :: \{\text{banach}, \text{second_countable_topology}\}$
assumes $a < b$
assumes $\text{approx: } \text{einterval } a\ b = (\bigcup i. \{l\ i .. u\ i\})$
 $\text{incseq } u\ \text{decseq } l \wedge i. l\ i < u\ i \wedge i. a < l\ i \wedge i. u\ i < b$
 $l \longrightarrow a\ u \longrightarrow b$
assumes $f_integrable: \text{set_integrable lborel } (\text{einterval } a\ b)\ f$
shows $(\lambda i. \text{LBINT } x=l\ i.. u\ i. f\ x) \longrightarrow (\text{LBINT } x=a..b. f\ x)$

9.10.15 A slightly stronger Fundamental Theorem of Calculus

theorem *interval_integral_FTC_integrable:*
fixes $f\ F :: \text{real} \Rightarrow 'a :: \text{euclidean_space}$ **and** $a\ b :: \text{ereal}$
assumes $a < b$
assumes $F: \bigwedge x. a < \text{ereal } x \Longrightarrow \text{ereal } x < b \Longrightarrow (F\ \text{has_vector_derivative } f\ x)$
(at x)
assumes $f: \bigwedge x. a < \text{ereal } x \Longrightarrow \text{ereal } x < b \Longrightarrow \text{isCont } f\ x$
assumes $f_integrable: \text{set_integrable lborel } (\text{einterval } a\ b)\ f$
assumes $A: ((F \circ \text{real_of_ereal}) \longrightarrow A)$ *(at_right a)*
assumes $B: ((F \circ \text{real_of_ereal}) \longrightarrow B)$ *(at_left b)*
shows $(\text{LBINT } x=a..b. f\ x) = B - A$

theorem *interval_integral_FTC2:*
fixes $a\ b\ c :: \text{real}$ **and** $f :: \text{real} \Rightarrow 'a :: \text{euclidean_space}$
assumes $a \leq c\ c \leq b$
and $\text{contf: continuous_on } \{a..b\}\ f$
fixes $x :: \text{real}$
assumes $a \leq x$ **and** $x \leq b$
shows $((\lambda u. \text{LBINT } y=c..u. f\ y)\ \text{has_vector_derivative } (f\ x))$ *(at x within {a..b})*

proposition *einterval_antiderivative:*
fixes $a\ b :: \text{ereal}$ **and** $f :: \text{real} \Rightarrow 'a :: \text{euclidean_space}$
assumes $a < b$ **and** $\text{contf: } \bigwedge x :: \text{real}. a < x \Longrightarrow x < b \Longrightarrow \text{isCont } f\ x$
shows $\exists F. \forall x :: \text{real}. a < x \longrightarrow x < b \longrightarrow (F\ \text{has_vector_derivative } f\ x)$ *(at x)*

9.10.16 The substitution theorem

theorem *interval_integral_substitution_finite:*
fixes $a\ b :: \text{real}$ **and** $f :: \text{real} \Rightarrow 'a :: \text{euclidean_space}$
assumes $a \leq b$
and $\text{derivg: } \bigwedge x. a \leq x \Longrightarrow x \leq b \Longrightarrow (g\ \text{has_real_derivative } (g'\ x))$ *(at x within {a..b})*

and *contf* : *continuous_on* (g ' {a..b}) f
and *contg'*: *continuous_on* {a..b} g'
shows (LBINT x=a..b. g' x *_R f (g x)) = (LBINT y=g a..g b. f y)

theorem *interval_integral_substitution_integrable*:

fixes f :: real \Rightarrow 'a::euclidean_space **and** a b u v :: ereal
assumes a < b
and *deriv_g*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{DERIV } g \ x \ :> \ g' \ x$
and *contf*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{isCont } f \ (g \ x)$
and *contg'*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{isCont } g' \ x$
and *g'_nonneg*: $\bigwedge x. a \leq \text{ereal } x \implies \text{ereal } x \leq b \implies 0 \leq g' \ x$
and A: ((ereal \circ g \circ real_of_ereal) \longrightarrow A) (at_right a)
and B: ((ereal \circ g \circ real_of_ereal) \longrightarrow B) (at_left b)
and *integrable*: set_integrable lborel (einterval a b) ($\lambda x. g' \ x \ *_{R} \ f \ (g \ x)$)
and *integrable2*: set_integrable lborel (einterval A B) ($\lambda x. f \ x$)
shows (LBINT x=A..B. f x) = (LBINT x=a..b. g' x *_R f (g x))

theorem *interval_integral_substitution_nonneg*:

fixes f g g':: real \Rightarrow real **and** a b u v :: ereal
assumes a < b
and *deriv_g*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{DERIV } g \ x \ :> \ g' \ x$
and *contf*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{isCont } f \ (g \ x)$
and *contg'*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies \text{isCont } g' \ x$
and *f_nonneg*: $\bigwedge x. a < \text{ereal } x \implies \text{ereal } x < b \implies 0 \leq f \ (g \ x)$
and *g'_nonneg*: $\bigwedge x. a \leq \text{ereal } x \implies \text{ereal } x \leq b \implies 0 \leq g' \ x$
and A: ((ereal \circ g \circ real_of_ereal) \longrightarrow A) (at_right a)
and B: ((ereal \circ g \circ real_of_ereal) \longrightarrow B) (at_left b)
and *integrable_fg*: set_integrable lborel (einterval a b) ($\lambda x. f \ (g \ x) \ * \ g' \ x$)
shows
 set_integrable lborel (einterval A B) f
 (LBINT x=A..B. f x) = (LBINT x=a..b. (f (g x) * g' x))

proposition *interval_integral_norm*:

fixes f :: real \Rightarrow 'a :: {banach, second_countable_topology}
shows interval_lebesgue_integrable lborel a b f \implies a \leq b \implies
 norm (LBINT t=a..b. f t) \leq LBINT t=a..b. norm (f t)

proposition *interval_integral_norm2*:

interval_lebesgue_integrable lborel a b f \implies
 norm (LBINT t=a..b. f t) \leq |LBINT t=a..b. norm (f t)|

end

9.11 Integration by Substitution for the Lebesgue Integral

theory *Lebesgue_Integral_Substitution*
imports *Interval_Integral*
begin

theorem *nn_integral_substitution*:

fixes $f :: \text{real} \Rightarrow \text{real}$
assumes $Mf[\text{measurable}]$: $\text{set_borel_measurable borel } \{g \ a..g \ b\} \ f$
assumes derivg : $\bigwedge x. x \in \{a..b\} \Longrightarrow (g \ \text{has_real_derivative} \ g' \ x) \ (at \ x)$
assumes contg' : $\text{continuous_on } \{a..b\} \ g'$
assumes derivg_nonneg : $\bigwedge x. x \in \{a..b\} \Longrightarrow g' \ x \geq 0$
assumes $a \leq b$
shows $(\int^{+x}. f \ x * \text{indicator } \{g \ a..g \ b\} \ x \ \partial \text{lborel}) =$
 $(\int^{+x}. f \ (g \ x) * g' \ x * \text{indicator } \{a..b\} \ x \ \partial \text{lborel})$

theorem *integral_substitution*:

assumes integrable : $\text{set_integrable lborel } \{g \ a..g \ b\} \ f$
assumes derivg : $\bigwedge x. x \in \{a..b\} \Longrightarrow (g \ \text{has_real_derivative} \ g' \ x) \ (at \ x)$
assumes contg' : $\text{continuous_on } \{a..b\} \ g'$
assumes derivg_nonneg : $\bigwedge x. x \in \{a..b\} \Longrightarrow g' \ x \geq 0$
assumes $a \leq b$
shows $\text{set_integrable lborel } \{a..b\} \ (\lambda x. f \ (g \ x) * g' \ x)$
and $(\text{LBINT } x. f \ x * \text{indicator } \{g \ a..g \ b\} \ x) = (\text{LBINT } x. f \ (g \ x) * g' \ x * \text{indicator } \{a..b\} \ x)$

theorem *interval_integral_substitution*:

assumes integrable : $\text{set_integrable lborel } \{g \ a..g \ b\} \ f$
assumes derivg : $\bigwedge x. x \in \{a..b\} \Longrightarrow (g \ \text{has_real_derivative} \ g' \ x) \ (at \ x)$
assumes contg' : $\text{continuous_on } \{a..b\} \ g'$
assumes derivg_nonneg : $\bigwedge x. x \in \{a..b\} \Longrightarrow g' \ x \geq 0$
assumes $a \leq b$
shows $\text{set_integrable lborel } \{a..b\} \ (\lambda x. f \ (g \ x) * g' \ x)$
and $(\text{LBINT } x=g \ a..g \ b. f \ x) = (\text{LBINT } x=a..b. f \ (g \ x) * g' \ x)$

end

9.12 The Volume of an n -Dimensional Ball

theory *Ball_Volume*

imports *Gamma_Function Lebesgue_Integral_Substitution*

begindefinition *unit_ball_vol* :: $\text{real} \Rightarrow \text{real}$ **where**

$\text{unit_ball_vol } n = \text{pi } \text{powr } (n / 2) / \text{Gamma } (n / 2 + 1)$

corollary *content_ball*:

$\text{content } (\text{ball } c \ r) = \text{unit_ball_vol } (\text{DIM}('a)) * r \wedge \text{DIM}('a)$

end

9.13 Integral Test for Summability

```

theory Integral_Test
imports Henstock_Kurzweil_Integration
beginlocale antimono_fun_sum_integral_diff =
  fixes  $f :: \text{real} \Rightarrow \text{real}$ 
  assumes dec:  $\bigwedge x y. x \geq 0 \implies x \leq y \implies f x \geq f y$ 
  assumes nonneg:  $\bigwedge x. x \geq 0 \implies f x \geq 0$ 
  assumes cont: continuous_on {0..}  $f$ 
begin

theorem integral_test:
  summable ( $\lambda n. f$  (of_nat  $n$ ))  $\longleftrightarrow$  convergent ( $\lambda n. \text{integral}$  {0..of_nat  $n$ }  $f$ )

end

```

9.14 Continuity of the indefinite integral; improper integral theorem

```

theory Improper_Integral
imports Equivalence_Lebesgue_Henstock_Integration
begin

```

9.14.1 Equiintegrability

```

definition equiintegrable_on (infixr equiintegrable'_on 46)
  where  $F$  equiintegrable_on  $I \equiv$ 
     $(\forall f \in F. f \text{ integrable\_on } I) \wedge$ 
     $(\forall e > 0. \exists \gamma. \text{gauge } \gamma \wedge$ 
       $(\forall f \mathcal{D}. f \in F \wedge \mathcal{D} \text{ tagged\_division\_of } I \wedge \gamma \text{ fine } \mathcal{D}$ 
         $\longrightarrow \text{norm } ((\sum (x,K) \in \mathcal{D}. \text{content } K *_R f x) - \text{integral } I f)$ 
         $< e))$ 

```

```

corollary equiintegrable_sum_real:
  fixes  $F :: (\text{real} \Rightarrow 'b :: \text{euclidean\_space}) \text{ set}$ 
  assumes  $F$  equiintegrable_on { $a..b$ }
  shows  $(\bigcup I \in \text{Collect finite}. \bigcup c \in \{c. (\forall i \in I. c i \geq 0) \wedge \text{sum } c I = 1\}.$ 
     $\bigcup f \in I \rightarrow F. \{(\lambda x. \text{sum } (\lambda i. c i *_R f i x) I)\})$ 
    equiintegrable_on { $a..b$ })
theorem equiintegrable_limit:
  fixes  $g :: 'a :: \text{euclidean\_space} \Rightarrow 'b :: \text{banach}$ 

```

assumes *feq*: *range f equiintegrable_on cbox a b*
and *to_g*: $\bigwedge x. x \in \text{cbox } a \ b \implies (\lambda n. f \ n \ x) \longrightarrow g \ x$
shows *g integrable_on cbox a b* $\wedge (\lambda n. \text{integral } (\text{cbox } a \ b) (f \ n)) \longrightarrow \text{integral } (\text{cbox } a \ b) \ g$

9.14.2 Subinterval restrictions for equiintegrable families

proposition *sum_content_area_over_thin_division*:

assumes *div*: \mathcal{D} *division_of S* **and** *S*: $S \subseteq \text{cbox } a \ b$ **and** *i*: $i \in \text{Basis}$
and $a \cdot i \leq c \leq b \cdot i$
and *nonmt*: $\bigwedge K. K \in \mathcal{D} \implies K \cap \{x. x \cdot i = c\} \neq \{\}$
shows $(b \cdot i - a \cdot i) * (\sum K \in \mathcal{D}. \text{content } K / (\text{interval_upperbound } K \cdot i - \text{interval_lowerbound } K \cdot i))$
 $\leq 2 * \text{content}(\text{cbox } a \ b)$

proposition *bounded_equiintegral_over_thin_tagged_partial_division*:

fixes *f* :: $'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *F*: F *equiintegrable_on cbox a b* **and** *f*: $f \in F$ **and** $0 < \varepsilon$
and *norm_f*: $\bigwedge h \ x. \llbracket h \in F; x \in \text{cbox } a \ b \rrbracket \implies \text{norm}(h \ x) \leq \text{norm}(f \ x)$
obtains γ **where** *gauge* γ
 $\bigwedge c \ i \ S \ h. \llbracket c \in \text{cbox } a \ b; i \in \text{Basis}; S \text{ tagged_partial_division_of } \text{cbox } a \ b;$
 $\gamma \text{ fine } S; h \in F; \bigwedge x \ K. (x, K) \in S \implies (K \cap \{x. x \cdot i = c \cdot i\}$
 $\neq \{\}) \rrbracket$
 $\implies (\sum (x, K) \in S. \text{norm } (\text{integral } K \ h)) < \varepsilon$

proposition *equiintegrable_halfspace_restrictions_le*:

fixes *f* :: $'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *F*: F *equiintegrable_on cbox a b* **and** *f*: $f \in F$
and *norm_f*: $\bigwedge h \ x. \llbracket h \in F; x \in \text{cbox } a \ b \rrbracket \implies \text{norm}(h \ x) \leq \text{norm}(f \ x)$
shows $(\bigcup i \in \text{Basis}. \bigcup c. \bigcup h \in F. \{(\lambda x. \text{if } x \cdot i \leq c \text{ then } h \ x \text{ else } 0)\})$
equiintegrable_on cbox a b

corollary *equiintegrable_halfspace_restrictions_ge*:

fixes *f* :: $'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes *F*: F *equiintegrable_on cbox a b* **and** *f*: $f \in F$
and *norm_f*: $\bigwedge h \ x. \llbracket h \in F; x \in \text{cbox } a \ b \rrbracket \implies \text{norm}(h \ x) \leq \text{norm}(f \ x)$
shows $(\bigcup i \in \text{Basis}. \bigcup c. \bigcup h \in F. \{(\lambda x. \text{if } x \cdot i \geq c \text{ then } h \ x \text{ else } 0)\})$
equiintegrable_on cbox a b

corollary *equiintegrable_halfspace_restrictions_lt*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $F: F \text{ equiintegrable_on cbox } a \ b$ **and** $f: f \in F$
and $\text{norm_f}: \bigwedge h \ x. \llbracket h \in F; x \in \text{cbox } a \ b \rrbracket \implies \text{norm}(h \ x) \leq \text{norm}(f \ x)$
shows $(\bigcup i \in \text{Basis}. \bigcup c. \bigcup h \in F. \{(\lambda x. \text{if } x \cdot i < c \text{ then } h \ x \text{ else } 0)\}) \text{ equiintegrable_on cbox } a \ b$
(is ?G equiintegrable_on cbox } a \ b)

corollary *equiintegrable_halfspace_restrictions_gt*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $F: F \text{ equiintegrable_on cbox } a \ b$ **and** $f: f \in F$
and $\text{norm_f}: \bigwedge h \ x. \llbracket h \in F; x \in \text{cbox } a \ b \rrbracket \implies \text{norm}(h \ x) \leq \text{norm}(f \ x)$
shows $(\bigcup i \in \text{Basis}. \bigcup c. \bigcup h \in F. \{(\lambda x. \text{if } x \cdot i > c \text{ then } h \ x \text{ else } 0)\}) \text{ equiintegrable_on cbox } a \ b$
(is ?G equiintegrable_on cbox } a \ b)

proposition *equiintegrable_closed_interval_restrictions*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $f: f \text{ integrable_on cbox } a \ b$
shows $(\bigcup c \ d. \{(\lambda x. \text{if } x \in \text{cbox } c \ d \text{ then } f \ x \text{ else } 0)\}) \text{ equiintegrable_on cbox } a \ b$

9.14.3 Continuity of the indefinite integral

proposition *indefinite_integral_continuous*:

fixes $f :: 'a :: \text{euclidean_space} \Rightarrow 'b :: \text{euclidean_space}$
assumes $\text{int_f}: f \text{ integrable_on cbox } a \ b$
and $c: c \in \text{cbox } a \ b$ **and** $d: d \in \text{cbox } a \ b$ $0 < \varepsilon$
obtains δ **where** $0 < \delta$
 $\bigwedge c' \ d'. \llbracket c' \in \text{cbox } a \ b; d' \in \text{cbox } a \ b; \text{norm}(c' - c) \leq \delta; \text{norm}(d' - d) \leq \delta \rrbracket$
 $\implies \text{norm}(\text{integral}(\text{cbox } c' \ d') \ f - \text{integral}(\text{cbox } c \ d) \ f) < \varepsilon$

corollary *indefinite_integral_uniformly_continuous*:

fixes $f :: 'a :: \text{euclidean_space} \Rightarrow 'b :: \text{euclidean_space}$
assumes $f \text{ integrable_on cbox } a \ b$
shows $\text{uniformly_continuous_on}(\text{cbox}(\text{Pair } a \ a) (\text{Pair } b \ b)) (\lambda y. \text{integral}(\text{cbox}(\text{fst } y) (\text{snd } y)) \ f)$

corollary *bounded_integrals_over_subintervals*:

fixes $f :: 'a :: \text{euclidean_space} \Rightarrow 'b :: \text{euclidean_space}$
assumes $f \text{ integrable_on cbox } a \ b$
shows $\text{bounded} \{\text{integral}(\text{cbox } c \ d) \ f \mid c \ d. \text{cbox } c \ d \subseteq \text{cbox } a \ b\}$

theorem *absolutely_integrable_improper*:

fixes $f :: 'M::\text{euclidean_space} \Rightarrow 'N::\text{euclidean_space}$
assumes $\text{int_f}: \bigwedge c \ d. \text{cbox } c \ d \subseteq \text{box } a \ b \implies f \text{ integrable_on cbox } c \ d$
and $\text{bo}: \text{bounded} \{\text{integral}(\text{cbox } c \ d) \ f \mid c \ d. \text{cbox } c \ d \subseteq \text{box } a \ b\}$
and $\text{absi}: \bigwedge i. i \in \text{Basis}$

$\implies \exists g. g \text{ absolutely_integrable_on } \text{cbox } a \ b \wedge$
 $(\forall x \in \text{cbox } a \ b. f \ x \cdot i \leq g \ x) \vee (\forall x \in \text{cbox } a \ b. f \ x \cdot i \geq g \ x)$
shows $f \text{ absolutely_integrable_on } \text{cbox } a \ b$

9.14.4 Second mean value theorem and corollaries

theorem *second_mean_value_theorem_full*:

fixes $f :: \text{real} \Rightarrow \text{real}$
assumes $f: f \text{ integrable_on } \{a..b\}$ **and** $a \leq b$
and $g: \bigwedge x \ y. \llbracket a \leq x; x \leq y; y \leq b \rrbracket \implies g \ x \leq g \ y$
obtains c **where** $c \in \{a..b\}$
and $((\lambda x. g \ x * f \ x) \text{ has_integral } (g \ a * \text{integral } \{a..c\} f + g \ b * \text{integral } \{c..b\} f)) \{a..b\}$

corollary *second_mean_value_theorem*:

fixes $f :: \text{real} \Rightarrow \text{real}$
assumes $f: f \text{ integrable_on } \{a..b\}$ **and** $a \leq b$
and $g: \bigwedge x \ y. \llbracket a \leq x; x \leq y; y \leq b \rrbracket \implies g \ x \leq g \ y$
obtains c **where** $c \in \{a..b\}$
 $\text{integral } \{a..b\} (\lambda x. g \ x * f \ x) = g \ a * \text{integral } \{a..c\} f + g \ b * \text{integral } \{c..b\} f$

end

9.15 Continuous Extensions of Functions

theory *Continuous_Extension*

imports *Starlike*

begin

9.15.1 Partitions of unity subordinate to locally finite open coverings

proposition *subordinate_partition_of_unity*:

fixes $S :: 'a::\text{metric_space set}$
assumes $S \subseteq \bigcup C$ **and** $opC: \bigwedge T. T \in C \implies \text{open } T$
and $fin: \bigwedge x. x \in S \implies \exists V. \text{open } V \wedge x \in V \wedge \text{finite } \{U \in C. U \cap V \neq \{\}\}$
obtains $F :: ['a \text{ set}, 'a] \Rightarrow \text{real}$
where $\bigwedge U. U \in C \implies \text{continuous_on } S (F \ U) \wedge (\forall x \in S. 0 \leq F \ U \ x)$
and $\bigwedge x \ U. \llbracket U \in C; x \in S; x \notin U \rrbracket \implies F \ U \ x = 0$
and $\bigwedge x. x \in S \implies \text{supp_sum } (\lambda W. F \ W \ x) \ C = 1$
and $\bigwedge x. x \in S \implies \exists V. \text{open } V \wedge x \in V \wedge \text{finite } \{U \in C. \exists x \in V. F \ U \ x \neq 0\}$

9.15.2 Urysohn's Lemma for Euclidean Spaces

proposition *Urysohn_local_strong*:
assumes $US: \text{closedin } (\text{top_of_set } U) S$
and $UT: \text{closedin } (\text{top_of_set } U) T$
and $S \cap T = \{\} \ a \neq b$
obtains $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
where $\text{continuous_on } U f$
 $\bigwedge x. x \in U \implies f x \in \text{closed_segment } a b$
 $\bigwedge x. x \in U \implies (f x = a \longleftrightarrow x \in S)$
 $\bigwedge x. x \in U \implies (f x = b \longleftrightarrow x \in T)$

proposition *Urysohn*:
assumes $US: \text{closed } S$
and $UT: \text{closed } T$
and $S \cap T = \{\}$
obtains $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
where $\text{continuous_on } UNIV f$
 $\bigwedge x. f x \in \text{closed_segment } a b$
 $\bigwedge x. x \in S \implies f x = a$
 $\bigwedge x. x \in T \implies f x = b$

9.15.3 Dugundji's Extension Theorem and Tietze Variants

theorem *Dugundji*:
fixes $f :: 'a::\{\text{metric_space}, \text{second_countable_topology}\} \Rightarrow 'b::\text{real_inner}$
assumes $\text{convex } C \ C \neq \{\}$
and $\text{cloin}: \text{closedin } (\text{top_of_set } U) S$
and $\text{contf}: \text{continuous_on } S f$ **and** $f ' S \subseteq C$
obtains g **where** $\text{continuous_on } U g \ g ' U \subseteq C$
 $\bigwedge x. x \in S \implies g x = f x$

corollary *Tietze*:
fixes $f :: 'a::\{\text{metric_space}, \text{second_countable_topology}\} \Rightarrow 'b::\text{real_inner}$
assumes $\text{continuous_on } S f$
and $\text{closedin } (\text{top_of_set } U) S$
and $0 \leq B$
and $\bigwedge x. x \in S \implies \text{norm}(f x) \leq B$
obtains g **where** $\text{continuous_on } U g \ \bigwedge x. x \in S \implies g x = f x$
 $\bigwedge x. x \in U \implies \text{norm}(g x) \leq B$

end

9.16 Equivalence Between Classical Borel Measurability and HOL Light's

```
theory Equivalence_Measurable_On_Borel
  imports Equivalence_Lebesgue_Henstock_Integration Improper_Integral Continuous_Extension
begin
```

9.16.1 Austin's Lemma

9.16.2 A differentiability-like property of the indefinite integral.

```
proposition integrable_ccontinuous_explicit:
  fixes f :: 'a::euclidean_space  $\Rightarrow$  'b::euclidean_space
  assumes  $\bigwedge a b::'a. f \text{ integrable\_on } \text{cbox } a b$ 
  obtains  $N$  where
    negligible  $N$ 
     $\bigwedge x e. \llbracket x \notin N; 0 < e \rrbracket \implies$ 
       $\exists d > 0. \forall h. 0 < h \wedge h < d \longrightarrow$ 
         $\text{norm}(\text{integral } (\text{cbox } x (x + h *_{\mathbb{R}} \text{One})) f /_{\mathbb{R}} h \wedge \text{DIM}('a) - f$ 
 $x) < e$ 
```

9.16.3 HOL Light measurability

```
proposition integrable_subintervals_imp_measurable:
  fixes f :: 'a::euclidean_space  $\Rightarrow$  'b::euclidean_space
  assumes  $\bigwedge a b. f \text{ integrable\_on } \text{cbox } a b$ 
  shows  $f \text{ measurable\_on } \text{UNIV}$ 
```

9.16.4 Composing continuous and measurable functions; a few variants

```
proposition indicator_measurable_on:
  assumes  $S \in \text{sets lebesgue}$ 
  shows  $\text{indicat\_real } S \text{ measurable\_on } \text{UNIV}$ 
```

```
lemma simple_function_induct_real
  [consumes 1, case_names cong set mult add, induct set: simple_function]:
  fixes  $u :: 'a \Rightarrow \text{real}$ 
  assumes  $u: \text{simple\_function } M u$ 
```

assumes cong: $\bigwedge f g. \text{simple_function } M f \implies \text{simple_function } M g \implies (AE x \text{ in } M. f x = g x) \implies P f \implies P g$
assumes set: $\bigwedge A. A \in \text{sets } M \implies P$ (indicator A)
assumes mult: $\bigwedge u c. P u \implies P (\lambda x. c * u x)$
assumes add: $\bigwedge u v. P u \implies P v \implies P (\lambda x. u x + v x)$
and nn: $\bigwedge x. u x \geq 0$
shows $P u$

proposition *simple_function_measurable_on_UNIV:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow \text{real}$
assumes f : *simple_function lebesgue* f **and** nn : $\bigwedge x. f x \geq 0$
shows f *measurable_on UNIV*

corollary *simple_function_measurable_on:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow \text{real}$
assumes f : *simple_function lebesgue* f **and** nn : $\bigwedge x. f x \geq 0$ **and** S : $S \in \text{sets lebesgue}$
shows f *measurable_on S*

proposition *measurable_on_componentwise_UNIV:*

f *measurable_on UNIV* $\longleftrightarrow (\forall i \in \text{Basis}. (\lambda x. (f x \cdot i) *_R i)$ *measurable_on UNIV*)
(is ?lhs = ?rhs)

corollary *measurable_on_componentwise:*

f *measurable_on S* $\longleftrightarrow (\forall i \in \text{Basis}. (\lambda x. (f x \cdot i) *_R i)$ *measurable_on S*)

lemma *borel_measurable_implies_simple_function_sequence_real:*

fixes $u :: 'a \Rightarrow \text{real}$
assumes u [*measurable*]: $u \in \text{borel_measurable } M$ **and** nn : $\bigwedge x. u x \geq 0$
shows $\exists f. \text{incseq } f \wedge (\forall i. \text{simple_function } M (f i)) \wedge (\forall x. \text{bdd_above } (\text{range } (\lambda i. f i x))) \wedge$
 $(\forall i x. 0 \leq f i x) \wedge u = (\text{SUP } i. f i)$

proposition *homeomorphic_box_UNIV:*

fixes $a b :: 'a::\text{euclidean_space}$
assumes $\text{box } a b \neq \{\}$
shows $\text{box } a b$ *homeomorphic* ($\text{UNIV}::'a \text{ set}$)

proposition *measurable_on_imp_borel_measurable_lebesgue_UNIV:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes f *measurable_on UNIV*
shows $f \in \text{borel_measurable lebesgue}$

corollary *measurable_on_imp_borel_measurable_lebesgue:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes f measurable_on S **and** $S: S \in sets\ lebesgue$
shows $f \in borel_measurable\ (lebesgue_on\ S)$

proposition *measurable_on_limit:*

fixes $f :: nat \Rightarrow 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $f: \bigwedge n. f\ n$ measurable_on S **and** $N: negligible\ N$
and $lim: \bigwedge x. x \in S - N \implies (\lambda n. f\ n\ x) \longrightarrow g\ x$
shows g measurable_on S

proposition *lebesgue_measurable_imp_measurable_on:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $f: f \in borel_measurable\ lebesgue$ **and** $S: S \in sets\ lebesgue$
shows f measurable_on S

proposition *measurable_on_iff_borel_measurable:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $S \in sets\ lebesgue$
shows f measurable_on $S \iff f \in borel_measurable\ (lebesgue_on\ S)$ (**is** ?lhs =
 ?rhs)

9.16.5 Monotonic functions are Lebesgue integrable

9.16.6 Measurability on generalisations of the binary product

end

9.17 Embedding Measure Spaces with a Function

theory *Embed_Measure*

imports *Binary_Product_Measure*

begindefinition *embed_measure* :: $'a\ measure \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'b\ measure$ **where**
 $embed_measure\ M\ f = measure_of\ (f\ 'space\ M)\ \{f\ 'A\ |A. A \in sets\ M\}$
 $(\lambda A. emeasure\ M\ (f\ -'A \cap space\ M))$

end

9.18 Brouwer's Fixed Point Theorem

```
theory Brouwer_Fixpoint
  imports Homeomorphism Derivative
begin
```

9.18.1 Retractions

9.18.2 Kuhn Simplices

9.18.3 Brouwer's fixed point theorem

```
theorem brouwer:
  fixes f :: 'a::euclidean_space  $\Rightarrow$  'a
  assumes S: compact S convex S S  $\neq$  {}
    and contf: continuous_on S f
    and fim: f  $\in$  S  $\rightarrow$  S
  obtains x where x  $\in$  S and f x = x
```

9.18.4 Applications

```
corollary no_retraction_cball:
  fixes a :: 'a::euclidean_space
  assumes e > 0
  shows  $\neg$  (frontier (cball a e) retract_of (cball a e))
```

```
corollary contractible_sphere:
  fixes a :: 'a::euclidean_space
  shows contractible(sphere a r)  $\longleftrightarrow$  r  $\leq$  0
```

```
corollary connected_sphere_eq:
  fixes a :: 'a :: euclidean_space
  shows connected(sphere a r)  $\longleftrightarrow$  2  $\leq$  DIM('a)  $\vee$  r  $\leq$  0
  (is ?lhs = ?rhs)
```

```
corollary path_connected_sphere_eq:
  fixes a :: 'a :: euclidean_space
  shows path_connected(sphere a r)  $\longleftrightarrow$  2  $\leq$  DIM('a)  $\vee$  r  $\leq$  0
  (is ?lhs = ?rhs)
```

```
proposition frontier_subset_retraction:
  fixes S :: 'a::euclidean_space set
  assumes bounded S and fros: frontier S  $\subseteq$  T
    and contf: continuous_on (closure S) f
    and fim: f  $\in$  S  $\rightarrow$  T
    and fid:  $\bigwedge x. x \in T \implies f x = x$ 
```

shows $S \subseteq T$

corollary *rel_frontier_retract_of_punctured_affine_hull:*

fixes $S :: 'a::euclidean_space\ set$

assumes *bounded S convex S a ∈ rel_interior S*

shows *rel_frontier S retract_of (affine hull S - {a})*

corollary *rel_boundary_retract_of_punctured_affine_hull:*

fixes $S :: 'a::euclidean_space\ set$

assumes *compact S convex S a ∈ rel_interior S*

shows *(S - rel_interior S) retract_of (affine hull S - {a})*

theorem *has_derivative_inverse_on:*

fixes $f :: 'n::euclidean_space \Rightarrow 'n$

assumes *open S*

and $\bigwedge x. x \in S \implies (f\ has_derivative\ f'(x))\ (at\ x)$

and $\bigwedge x. x \in S \implies g\ (f\ x) = x$

and $f' x \circ g' x = id$

and $x \in S$

shows *(g has_derivative g'(x)) (at (f x))*

end

9.19 Fashoda Meet Theorem

theory *Fashoda_Theorem*

imports *Brouwer_Fixpoint Path_Connected Cartesian_Euclidean_Space*

begin

9.19.1 Bijections between intervals

definition *interval_bij* :: $'a \times 'a \Rightarrow 'a \times 'a \Rightarrow 'a \Rightarrow 'a::euclidean_space$

where *interval_bij* =

$(\lambda(a, b) (u, v) x. (\sum_{i \in Basis. (u \cdot i + (x \cdot i - a \cdot i) / (b \cdot i - a \cdot i) * (v \cdot i - u \cdot i))$
*_R i))

9.19.2 Fashoda meet theorem

proposition *fashoda_unit:*

fixes $f\ g :: real \Rightarrow real^2$

assumes $f\ ' \{-1 .. 1\} \subseteq cbox\ (-1)\ 1$

and $g\ ' \{-1 .. 1\} \subseteq cbox\ (-1)\ 1$

and *continuous_on* $\{-1 .. 1\}$ f

and *continuous_on* $\{-1 .. 1\}$ g

and $f\ (-1)\ \$1 = -1$

and $f\ 1\ \$1 = 1\ g\ (-1)\ \$2 = -1$

and $g\ 1\ \$2 = 1$

shows $\exists s \in \{-1 .. 1\}. \exists t \in \{-1 .. 1\}. f s = g t$

proposition *fashoda_unit_path*:

fixes $f g :: \text{real} \Rightarrow \text{real}^2$

assumes *path* f

and *path* g

and $\text{path_image } f \subseteq \text{cbox } (-1) 1$

and $\text{path_image } g \subseteq \text{cbox } (-1) 1$

and $(\text{pathstart } f)\$1 = -1$

and $(\text{pathfinish } f)\$1 = 1$

and $(\text{pathstart } g)\$2 = -1$

and $(\text{pathfinish } g)\$2 = 1$

obtains z where $z \in \text{path_image } f$ and $z \in \text{path_image } g$

theorem *fashoda*:

fixes $b :: \text{real}^2$

assumes *path* f

and *path* g

and $\text{path_image } f \subseteq \text{cbox } a b$

and $\text{path_image } g \subseteq \text{cbox } a b$

and $(\text{pathstart } f)\$1 = a\1

and $(\text{pathfinish } f)\$1 = b\1

and $(\text{pathstart } g)\$2 = a\2

and $(\text{pathfinish } g)\$2 = b\2

obtains z where $z \in \text{path_image } f$ and $z \in \text{path_image } g$

9.19.3 Useful Fashoda corollary pointed out to me by Tom Hales

corollary *fashoda_interlace*:

fixes $a :: \text{real}^2$

assumes *path* f

and *path* g

and $\text{paf: path_image } f \subseteq \text{cbox } a b$

and $\text{pag: path_image } g \subseteq \text{cbox } a b$

and $(\text{pathstart } f)\$2 = a\2

and $(\text{pathfinish } f)\$2 = a\2

and $(\text{pathstart } g)\$2 = a\2

and $(\text{pathfinish } g)\$2 = a\2

and $(\text{pathstart } f)\$1 < (\text{pathstart } g)\1

and $(\text{pathstart } g)\$1 < (\text{pathfinish } f)\1

and $(\text{pathfinish } f)\$1 < (\text{pathfinish } g)\1

obtains z where $z \in \text{path_image } f$ and $z \in \text{path_image } g$

end

9.20 Vector Cross Products in 3 Dimensions

theory *Cross3*

imports *Determinants Cartesian_Euclidean_Space*

begin

definition *cross3* :: $[real^3, real^3] \Rightarrow real^3$ (**infixr** \times 80)

where $a \times b \equiv$

$$\text{vector } [a\$2 * b\$3 - a\$3 * b\$2, \\ a\$3 * b\$1 - a\$1 * b\$3, \\ a\$1 * b\$2 - a\$2 * b\$1]$$

9.20.1 Basic lemmas

proposition *Jacobi*: $x \times (y \times z) + y \times (z \times x) + z \times (x \times y) = 0$ **for** $x::real^3$

proposition *Lagrange*: $x \times (y \times z) = (x \cdot z) *_R y - (x \cdot y) *_R z$

proposition *cross_triple*: $(x \times y) \cdot z = (y \times z) \cdot x$

proposition *dot_cross*: $(w \times x) \cdot (y \times z) = (w \cdot y) * (x \cdot z) - (w \cdot z) * (x \cdot y)$

proposition *norm_cross*: $(\text{norm } (x \times y))^2 = (\text{norm } x)^2 * (\text{norm } y)^2 - (x \cdot y)^2$

9.20.2 Preservation by rotation, or other orthogonal transformation up to sign

9.20.3 Continuity

end

9.21 Bounded Continuous Functions

theory *Bounded_Continuous_Function*

imports

Topology_Euclidean_Space

Uniform_Limit

begin

9.21.1 Definition

definition *bcontfun* = $\{f. \text{continuous_on UNIV } f \wedge \text{bounded } (\text{range } f)\}$

instantiation *bcontfun* :: $(\text{topological_space}, \text{metric_space}) \text{ metric_space}$

begin

lift_definition *dist_bcontfun* :: 'a \Rightarrow_C 'b \Rightarrow 'a \Rightarrow_C 'b \Rightarrow real
is $\lambda f g. (SUP x. dist (f x) (g x))$

9.21.2 Complete Space

instance *bcontfun* :: (metric_space, complete_space) complete_space

end

9.22 Infinite Products

theory *Infinite_Products*

imports *Topology_Euclidean_Space Complex_Transcendental*

begin

9.22.1 Definitions and basic properties

definition *raw_has_prod* :: [nat \Rightarrow 'a::{t2_space, comm_semiring_1}, nat, 'a]
 \Rightarrow bool

where *raw_has_prod* $f M p \equiv (\lambda n. \prod_{i \leq n.} f (i+M)) \longrightarrow p \wedge p \neq 0$

definition

has_prod :: (nat \Rightarrow 'a::{t2_space, comm_semiring_1}) \Rightarrow 'a \Rightarrow bool (**infix**
has'_prod 80)

where *f has_prod* $p \equiv raw_has_prod f 0 p \vee (\exists i q. p = 0 \wedge f i = 0 \wedge$
raw_has_prod $f (Suc i) q)$

definition *convergent_prod* :: (nat \Rightarrow 'a :: {t2_space, comm_semiring_1}) \Rightarrow bool

where

convergent_prod $f \equiv \exists M p. raw_has_prod f M p$

definition *prodinf* :: (nat \Rightarrow 'a::{t2_space, comm_semiring_1}) \Rightarrow 'a

(**binder** \prod 10)

where *prodinf* $f = (THE p. f has_prod p)$

9.22.2 Absolutely convergent products

definition *abs_convergent_prod* :: (nat \Rightarrow _) \Rightarrow bool **where**

abs_convergent_prod $f \longleftrightarrow convergent_prod (\lambda i. 1 + norm (f i - 1))$

lemma *convergent_prod_iff_convergent*:

fixes $f :: nat \Rightarrow 'a :: \{topological_semigroup_mult, t2_space, idom\}$

assumes $\bigwedge i. f i \neq 0$

shows *convergent_prod* $f \longleftrightarrow convergent (\lambda n. \prod_{i \leq n.} f i) \wedge lim (\lambda n. \prod_{i \leq n.} f$
 $i) \neq 0$

theorem *abs_convergent_prod_conv_summable*:
fixes $f :: \text{nat} \Rightarrow 'a :: \text{real_normed_div_algebra}$
shows $\text{abs_convergent_prod } f \longleftrightarrow \text{summable } (\lambda i. \text{norm } (f\ i - 1))$

9.22.3 More elementary properties

theorem *abs_convergent_prod_imp_convergent_prod*:
fixes $f :: \text{nat} \Rightarrow 'a :: \{\text{real_normed_div_algebra}, \text{complete_space}, \text{comm_ring_1}\}$
assumes $\text{abs_convergent_prod } f$
shows $\text{convergent_prod } f$

corollary *convergent_prod_offset_0*:
fixes $f :: \text{nat} \Rightarrow 'a :: \{\text{idom}, \text{topological_semigroup_mult}, \text{t2_space}\}$
assumes $\text{convergent_prod } f \wedge i. f\ i \neq 0$
shows $\exists p. \text{raw_has_prod } f\ 0\ p$

theorem *has_prod_iff*: $f\ \text{has_prod } x \longleftrightarrow \text{convergent_prod } f \wedge \text{prodinf } f = x$

9.22.4 Exponentials and logarithms

theorem *convergent_prod_iff_summable_real*:
fixes $a :: \text{nat} \Rightarrow \text{real}$
assumes $\bigwedge n. a\ n > 0$
shows $\text{convergent_prod } (\lambda k. 1 + a\ k) \longleftrightarrow \text{summable } a$ (**is** $?lhs = ?rhs$)

theorem *Ln_producing_complex*:
fixes $z :: \text{nat} \Rightarrow \text{complex}$
assumes $z: \bigwedge j. z\ j \neq 0$ **and** $\xi: \xi \neq 0$
shows $((\lambda n. \prod_{j \leq n} z\ j) \longrightarrow \xi) \longleftrightarrow (\exists k. (\lambda n. (\sum_{j \leq n} \text{Ln } (z\ j))) \longrightarrow \text{Ln } \xi + \text{of_int } k * (\text{of_real}(2 * \pi) * i))$ (**is** $?lhs = ?rhs$)

proposition *convergent_prod_iff_summable_complex*:
fixes $z :: \text{nat} \Rightarrow \text{complex}$
assumes $\bigwedge k. z\ k \neq 0$
shows $\text{convergent_prod } (\lambda k. z\ k) \longleftrightarrow \text{summable } (\lambda k. \text{Ln } (z\ k))$ (**is** $?lhs = ?rhs$)

proposition *summable_imp_convergent_prod_complex*:
fixes $z :: \text{nat} \Rightarrow \text{complex}$
assumes $z: \text{summable } (\lambda k. \text{norm } (z\ k))$ **and** $\text{non0}: \bigwedge k. z\ k \neq -1$
shows $\text{convergent_prod } (\lambda k. 1 + z\ k)$

corollary *summable_imp_convergent_prod_real*:
fixes $z :: \text{nat} \Rightarrow \text{real}$
assumes $z: \text{summable } (\lambda k. |z\ k|)$ **and** $\text{non0}: \bigwedge k. z\ k \neq -1$
shows $\text{convergent_prod } (\lambda k. 1 + z\ k)$

end

9.23 Sums over Infinite Sets

```
theory Infinite_Set_Sum
  imports Set_Integral Infinite_Sum
begin
```

```
definition abs_summable_on ::
  ('a  $\Rightarrow$  'b :: {banach, second_countable_topology})  $\Rightarrow$  'a set  $\Rightarrow$  bool
  (infix abs'_summable'_on 50)
where
  f abs_summable_on A  $\longleftrightarrow$  integrable (count_space A) f
```

```
definition infsetsum ::
  ('a  $\Rightarrow$  'b :: {banach, second_countable_topology})  $\Rightarrow$  'a set  $\Rightarrow$  'b
where
  infsetsum f A = lebesgue_integral (count_space A) f
```

```
theorem infsetsum_reindex:
  assumes inj_on g A
  shows infsetsum f (g ' A) = infsetsum ( $\lambda$ x. f (g x)) A
```

```
theorem infsetsum_Sigma:
  fixes A :: 'a set and B :: 'a  $\Rightarrow$  'b set
  assumes [simp]: countable A and  $\bigwedge$ i. countable (B i)
  assumes summable: f abs_summable_on (Sigma A B)
  shows infsetsum f (Sigma A B) = infsetsum ( $\lambda$ x. infsetsum ( $\lambda$ y. f (x, y)) (B x)) A
```

```
theorem abs_summable_on_Sigma_iff:
  assumes [simp]: countable A and  $\bigwedge$ x. x  $\in$  A  $\Longrightarrow$  countable (B x)
  shows f abs_summable_on Sigma A B  $\longleftrightarrow$ 
    ( $\forall$ x $\in$ A. ( $\lambda$ y. f (x, y)) abs_summable_on B x)  $\wedge$ 
    (( $\lambda$ x. infsetsum ( $\lambda$ y. norm (f (x, y))) (B x)) abs_summable_on A)
```

```
theorem infsetsum_prod_PiE:
  fixes f :: 'a  $\Rightarrow$  'b  $\Rightarrow$  'c :: {real_normed_field, banach, second_countable_topology}
  assumes finite: finite A and countable:  $\bigwedge$ x. x  $\in$  A  $\Longrightarrow$  countable (B x)
  assumes summable:  $\bigwedge$ x. x  $\in$  A  $\Longrightarrow$  f x abs_summable_on B x
  shows infsetsum ( $\lambda$ g.  $\prod$  x $\in$ A. f x (g x)) (PiE A B) = ( $\prod$  x $\in$ A. infsetsum (f x) (B x))
```


end

9.24 Faces, Extreme Points, Polytopes, Polyhedra etc

```
theory Polytope
imports Cartesian_Euclidean_Space Path_Connected
begin
```

9.24.1 Faces of a (usually convex) set

```
definition face_of :: ['a::real_vector set, 'a set]  $\Rightarrow$  bool (infixr (face'_of) 50)
where
  T face_of S  $\longleftrightarrow$ 
    T  $\subseteq$  S  $\wedge$  convex T  $\wedge$ 
    ( $\forall a \in S. \forall b \in S. \forall x \in T. x \in \text{open\_segment } a \ b \longrightarrow a \in T \wedge b \in T$ )
```

```
proposition face_of_imp_eq_affine_Int:
  fixes S :: 'a::euclidean_space set
  assumes S: convex S and T: T face_of S
  shows T = (affine hull T)  $\cap$  S
```

```
proposition face_of_conic:
  assumes conic S f face_of S
  shows conic f
```

```
proposition face_of_convex_hulls:
  assumes S: finite S T  $\subseteq$  S and disj: affine hull T  $\cap$  convex hull (S - T) =
  {}
  shows (convex hull T) face_of (convex hull S)
```

```
proposition face_of_convex_hull_insert:
  assumes finite S a  $\notin$  affine hull S and T: T face_of convex hull S
  shows T face_of convex hull insert a S
```

```
proposition face_of_affine_trivial:
  assumes affine S T face_of S
  shows T = {}  $\vee$  T = S
```

```
proposition Inter_faces_finite_altbound:
  fixes T :: 'a::euclidean_space set set
  assumes cfaI:  $\bigwedge c. c \in T \implies c \text{ face\_of } S$ 
  shows  $\exists F'. \text{finite } F' \wedge F' \subseteq T \wedge \text{card } F' \leq \text{DIM}('a) + 2 \wedge \bigcap F' = \bigcap T$ 
```

proposition *face_of_Times:*

assumes F *face_of* S **and** F' *face_of* S'
shows $(F \times F')$ *face_of* $(S \times S')$

corollary *face_of_Times_decomp:*

fixes $S :: 'a::euclidean_space$ *set* **and** $S' :: 'b::euclidean_space$ *set*
shows C *face_of* $(S \times S')$ \longleftrightarrow $(\exists F F'. F$ *face_of* $S \wedge F'$ *face_of* $S' \wedge C =$
 $F \times F')$
(is $?lhs = ?rhs)$

9.24.2 Exposed faces

definition *exposed_face_of* :: $[a::euclidean_space$ *set*, $'a$ *set*] \Rightarrow *bool*

(infixr $(exposed_face_of)$ 50)

where T *exposed_face_of* $S \longleftrightarrow$
 T *face_of* $S \wedge (\exists a b. S \subseteq \{x. a \cdot x \leq b\} \wedge T = S \cap \{x. a \cdot x = b\})$

proposition *exposed_face_of_Int:*

assumes T *exposed_face_of* S
and U *exposed_face_of* S
shows $(T \cap U)$ *exposed_face_of* S

proposition *exposed_face_of_Inter:*

fixes $P :: 'a::euclidean_space$ *set set*
assumes $P \neq \{\}$
and $\bigwedge T. T \in P \Longrightarrow T$ *exposed_face_of* S
shows $\bigcap P$ *exposed_face_of* S

proposition *exposed_face_of_sums:*

assumes *convex* S **and** *convex* T
and F *exposed_face_of* $\{x + y \mid x y. x \in S \wedge y \in T\}$
(is F *exposed_face_of* $?ST)$

obtains $k l$

where k *exposed_face_of* S l *exposed_face_of* T
 $F = \{x + y \mid x y. x \in k \wedge y \in l\}$

proposition *exposed_face_of_parallel:*

T *exposed_face_of* $S \longleftrightarrow$

T *face_of* $S \wedge$

$(\exists a b. S \subseteq \{x. a \cdot x \leq b\} \wedge T = S \cap \{x. a \cdot x = b\} \wedge$

$(T \neq \{\} \longrightarrow T \neq S \longrightarrow a \neq 0) \wedge$

$(T \neq S \longrightarrow (\forall w \in \text{affine hull } S. (w + a) \in \text{affine hull } S)))$

(is $?lhs = ?rhs)$

9.24.3 Extreme points of a set: its singleton faces

definition *extreme_point_of* :: $[a::real_vector$, $'a$ *set*] \Rightarrow *bool*

(**infixr** (*extreme'_point'_of*) 50)
where x *extreme_point_of* $S \longleftrightarrow$
 $x \in S \wedge (\forall a \in S. \forall b \in S. x \notin \text{open_segment } a \ b)$

proposition *extreme_points_of_convex_hull*:
 $\{x. x \text{ extreme_point_of } (\text{convex hull } S)\} \subseteq S$

9.24.4 Facets

definition *facet_of* :: [$'a::\text{euclidean_space set}$, $'a \text{ set}$] \Rightarrow *bool*
(**infixr** (*facet'_of*) 50)
where F *facet_of* $S \longleftrightarrow F$ *face_of* $S \wedge F \neq \{\} \wedge \text{aff_dim } F = \text{aff_dim } S - 1$

9.24.5 Edges: faces of affine dimension 1

definition *edge_of* :: [$'a::\text{euclidean_space set}$, $'a \text{ set}$] \Rightarrow *bool* (**infixr** (*edge'_of*)
50)
where e *edge_of* $S \longleftrightarrow e$ *face_of* $S \wedge \text{aff_dim } e = 1$

9.24.6 Existence of extreme points

proposition *different_norm_3_collinear_points*:
fixes $a :: 'a::\text{euclidean_space}$
assumes $x \in \text{open_segment } a \ b$ $\text{norm}(a) = \text{norm}(b)$ $\text{norm}(x) = \text{norm}(b)$
shows *False*

proposition *extreme_point_exists_convex*:
fixes $S :: 'a::\text{euclidean_space set}$
assumes $\text{compact } S$ $\text{convex } S$ $S \neq \{\}$
obtains x **where** x *extreme_point_of* S

9.24.7 Krein-Milman, the weaker form

proposition *Krein_Milman*:
fixes $S :: 'a::\text{euclidean_space set}$
assumes $\text{compact } S$ $\text{convex } S$
shows $S = \text{closure}(\text{convex hull } \{x. x \text{ extreme_point_of } S\})$

theorem *Krein_Milman_Minkowski*:
fixes $S :: 'a::\text{euclidean_space set}$
assumes $\text{compact } S$ $\text{convex } S$
shows $S = \text{convex hull } \{x. x \text{ extreme_point_of } S\}$

9.24.8 Applying it to convex hulls of explicitly indicated finite sets

corollary *Krein_Milman_polytope*:

fixes $S :: 'a :: euclidean_space$ set

shows

$finite\ S$

$\implies convex\ hull\ S =$

$convex\ hull\ \{x. x\ extreme_point_of\ (convex\ hull\ S)\}$

proposition *face_of_convex_hull_insert_eq*:

fixes $a :: 'a :: euclidean_space$

assumes $finite\ S$ and $a \notin affine\ hull\ S$

shows $(F\ face_of\ (convex\ hull\ (insert\ a\ S))) \longleftrightarrow$

$F\ face_of\ (convex\ hull\ S) \vee$

$(\exists F'. F'\ face_of\ (convex\ hull\ S) \wedge F = convex\ hull\ (insert\ a\ F'))$

(is $F\ face_of\ ?CAS \longleftrightarrow _)$

proposition *face_of_convex_hull_affine_independent*:

fixes $S :: 'a :: euclidean_space$ set

assumes $\neg affine_dependent\ S$

shows $(T\ face_of\ (convex\ hull\ S) \longleftrightarrow (\exists c. c \subseteq S \wedge T = convex\ hull\ c))$

(is $?lhs = ?rhs)$

proposition *Krein_Milman_frontier*:

fixes $S :: 'a :: euclidean_space$ set

assumes $convex\ S$ compact S

shows $S = convex\ hull\ (frontier\ S)$

(is $?lhs = ?rhs)$

9.24.9 Polytopes

definition *polytope where*

$polytope\ S \equiv \exists v. finite\ v \wedge S = convex\ hull\ v$

proposition *face_of_polytope_insert2*:

fixes $a :: 'a :: euclidean_space$

assumes $polytope\ S$ $a \notin affine\ hull\ S$ $F\ face_of\ S$

shows $convex\ hull\ (insert\ a\ F)\ face_of\ convex\ hull\ (insert\ a\ S)$

9.24.10 Polyhedra

definition *polyhedron where*

$polyhedron\ S \equiv$

$\exists F. finite\ F \wedge$

$$S = \bigcap F \wedge (\forall h \in F. \exists a b. a \neq 0 \wedge h = \{x. a \cdot x \leq b\})$$

9.24.11 Canonical polyhedron representation making facial structure explicit

proposition *polyhedron_Int_affine*:

fixes $S :: 'a :: \text{euclidean_space set}$

shows $\text{polyhedron } S \longleftrightarrow$

$$(\exists F. \text{finite } F \wedge S = (\text{affine hull } S) \cap \bigcap F \wedge (\forall h \in F. \exists a b. a \neq 0 \wedge h = \{x. a \cdot x \leq b\}))$$

proposition *rel_interior_polyhedron_explicit*:

assumes *finite F*

and seq: $S = \text{affine hull } S \cap \bigcap F$

and faceq: $\bigwedge h. h \in F \implies a h \neq 0 \wedge h = \{x. a h \cdot x \leq b h\}$

and psub: $\bigwedge F'. F' \subset F \implies S \subset \text{affine hull } S \cap \bigcap F'$

shows $\text{rel_interior } S = \{x \in S. \forall h \in F. a h \cdot x < b h\}$

proposition *polyhedron_Int_affine_parallel_minimal*:

fixes $S :: 'a :: \text{euclidean_space set}$

shows $\text{polyhedron } S \longleftrightarrow$

$$(\exists F. \text{finite } F \wedge S = (\text{affine hull } S) \cap (\bigcap F) \wedge (\forall h \in F. \exists a b. a \neq 0 \wedge h = \{x. a \cdot x \leq b\} \wedge (\forall x \in \text{affine hull } S. (x + a) \in \text{affine hull } S)) \wedge (\forall F'. F' \subset F \longrightarrow S \subset (\text{affine hull } S) \cap (\bigcap F')))$$

(is ?lhs = ?rhs)

proposition *facet_of_polyhedron_explicit*:

assumes *finite F*

and seq: $S = \text{affine hull } S \cap \bigcap F$

and faceq: $\bigwedge h. h \in F \implies a h \neq 0 \wedge h = \{x. a h \cdot x \leq b h\}$

and psub: $\bigwedge F'. F' \subset F \implies S \subset \text{affine hull } S \cap \bigcap F'$

shows $C \text{ facet_of } S \longleftrightarrow (\exists h. h \in F \wedge C = S \cap \{x. a h \cdot x = b h\})$

proposition *face_of_polyhedron_explicit*:

fixes $S :: 'a :: \text{euclidean_space set}$

assumes *finite F*

and seq: $S = \text{affine hull } S \cap \bigcap F$

and faceq: $\bigwedge h. h \in F \implies a h \neq 0 \wedge h = \{x. a h \cdot x \leq b h\}$

and psub: $\bigwedge F'. F' \subset F \implies S \subset \text{affine hull } S \cap \bigcap F'$

and C: $C \text{ face_of } S \text{ and } C \neq \{\}$ $C \neq S$

shows $C = \bigcap \{S \cap \{x. a h \cdot x = b h\} \mid h. h \in F \wedge C \subseteq S \cap \{x. a h \cdot x = b h\}\}$

$h\}}\}$

9.24.12 More general corollaries from the explicit representation

corollary *facet_of_polyhedron*:

assumes *polyhedron* S **and** C *facet_of* S

obtains a b **where** $a \neq 0$ $S \subseteq \{x. a \cdot x \leq b\}$ $C = S \cap \{x. a \cdot x = b\}$

corollary *face_of_polyhedron*:

assumes *polyhedron* S **and** C *face_of* S **and** $C \neq \{\}$ **and** $C \neq S$

shows $C = \bigcap \{F. F \text{ facet_of } S \wedge C \subseteq F\}$

proposition *rel_interior_of_polyhedron*:

fixes $S :: 'a :: \text{euclidean_space set}$

assumes *polyhedron* S

shows $\text{rel_interior } S = S - \bigcup \{F. F \text{ facet_of } S\}$

proposition *polyhedron_eq_finite_exposed_faces*:

fixes $S :: 'a :: \text{euclidean_space set}$

shows *polyhedron* $S \iff \text{closed } S \wedge \text{convex } S \wedge \text{finite } \{F. F \text{ exposed_face_of } S\}$

(**is** $?lhs = ?rhs$)

corollary *polyhedron_eq_finite_faces*:

fixes $S :: 'a :: \text{euclidean_space set}$

shows *polyhedron* $S \iff \text{closed } S \wedge \text{convex } S \wedge \text{finite } \{F. F \text{ face_of } S\}$

(**is** $?lhs = ?rhs$)

9.24.13 Relation between polytopes and polyhedra

proposition *polytope_eq_bounded_polyhedron*:

fixes $S :: 'a :: \text{euclidean_space set}$

shows *polytope* $S \iff \text{polyhedron } S \wedge \text{bounded } S$

(**is** $?lhs = ?rhs$)

9.24.14 Relative and absolute frontier of a polytope

proposition *frontier_of_convex_hull*:

fixes $S :: 'a :: \text{euclidean_space set}$

assumes $\text{card } S = \text{Suc } (\text{DIM } ('a))$

shows $\text{frontier } (\text{convex hull } S) = \bigcup \{\text{convex hull } (S - \{a\}) \mid a. a \in S\}$

9.24.15 Special case of a triangle

proposition *frontier_of_triangle*:

fixes $a :: 'a::euclidean_space$

assumes $DIM('a) = 2$

shows $frontier(convex\ hull\ \{a,b,c\}) = closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a$

(**is** ?lhs = ?rhs)

corollary *inside_of_triangle*:

fixes $a :: 'a::euclidean_space$

assumes $DIM('a) = 2$

shows $inside\ (closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a) = interior(convex\ hull\ \{a,b,c\})$

corollary *interior_of_triangle*:

fixes $a :: 'a::euclidean_space$

assumes $DIM('a) = 2$

shows $interior(convex\ hull\ \{a,b,c\}) = convex\ hull\ \{a,b,c\} - (closed_segment\ a\ b \cup closed_segment\ b\ c \cup closed_segment\ c\ a)$

9.24.16 Subdividing a cell complex

proposition *cell_complex_subdivision_exists*:

fixes $\mathcal{F} :: 'a::euclidean_space\ set\ set$

assumes $0 < e\ finite\ \mathcal{F}$

and *poly*: $\bigwedge X. X \in \mathcal{F} \implies polytope\ X$

and *aff*: $\bigwedge X. X \in \mathcal{F} \implies aff_dim\ X \leq d$

and *face*: $\bigwedge X\ Y. \llbracket X \in \mathcal{F}; Y \in \mathcal{F} \rrbracket \implies X \cap Y\ face_of\ X$

obtains \mathcal{F}' **where** *finite* $\mathcal{F}' \cup \mathcal{F}' = \bigcup \mathcal{F} \bigwedge X. X \in \mathcal{F}' \implies diameter\ X < e$

$\bigwedge X. X \in \mathcal{F}' \implies polytope\ X \bigwedge X. X \in \mathcal{F}' \implies aff_dim\ X \leq d$

$\bigwedge X\ Y. \llbracket X \in \mathcal{F}'; Y \in \mathcal{F}' \rrbracket \implies X \cap Y\ face_of\ X$

$\bigwedge C. C \in \mathcal{F}' \implies \exists D. D \in \mathcal{F} \wedge C \subseteq D$

$\bigwedge C\ x. C \in \mathcal{F} \wedge x \in C \implies \exists D. D \in \mathcal{F}' \wedge x \in D \wedge D \subseteq C$

9.24.17 Simplexes

definition *simplex* :: $int \Rightarrow 'a::euclidean_space\ set \Rightarrow bool$ (**infix** *simplex* 50)

where $n\ simplex\ S \equiv \exists C. \neg\ affine_dependent\ C \wedge int(card\ C) = n + 1 \wedge S = convex\ hull\ C$

9.24.18 Simplicial complexes and triangulations

definition *simplicial_complex* where

simplicial_complex $\mathcal{C} \equiv$
 $finite\ \mathcal{C} \wedge$
 $(\forall S \in \mathcal{C}. \exists n. n\ simplex\ S) \wedge$
 $(\forall F\ S. S \in \mathcal{C} \wedge F\ face_of\ S \longrightarrow F \in \mathcal{C}) \wedge$
 $(\forall S\ S'. S \in \mathcal{C} \wedge S' \in \mathcal{C} \longrightarrow (S \cap S')\ face_of\ S)$

definition *triangulation* where

triangulation $\mathcal{T} \equiv$
 $finite\ \mathcal{T} \wedge$
 $(\forall T \in \mathcal{T}. \exists n. n\ simplex\ T) \wedge$
 $(\forall T\ T'. T \in \mathcal{T} \wedge T' \in \mathcal{T} \longrightarrow (T \cap T')\ face_of\ T)$

9.24.19 Refining a cell complex to a simplicial complex

proposition *convex_hull_insert_Int_eq*:

fixes $z :: 'a :: euclidean_space$

assumes $z: z \in rel_interior\ S$

and $T: T \subseteq rel_frontier\ S$

and $U: U \subseteq rel_frontier\ S$

and $convex\ S\ convex\ T\ convex\ U$

shows $convex\ hull\ (insert\ z\ T) \cap convex\ hull\ (insert\ z\ U) = convex\ hull\ (insert\ z\ (T \cap U))$

(is ?lhs = ?rhs)

proposition *simplicial_subdivision_of_cell_complex*:

assumes $finite\ \mathcal{M}$

and $poly: \bigwedge C. C \in \mathcal{M} \implies polytope\ C$

and $face: \bigwedge C1\ C2. \llbracket C1 \in \mathcal{M}; C2 \in \mathcal{M} \rrbracket \implies C1 \cap C2\ face_of\ C1$

obtains \mathcal{T} where *simplicial_complex* \mathcal{T}

$\bigcup \mathcal{T} = \bigcup \mathcal{M}$

$\bigwedge C. C \in \mathcal{M} \implies \exists F. finite\ F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$

$\bigwedge K. K \in \mathcal{T} \implies \exists C. C \in \mathcal{M} \wedge K \subseteq C$

corollary *fine_simplicial_subdivision_of_cell_complex*:

assumes $0 < e\ finite\ \mathcal{M}$

and $poly: \bigwedge C. C \in \mathcal{M} \implies polytope\ C$

and $face: \bigwedge C1\ C2. \llbracket C1 \in \mathcal{M}; C2 \in \mathcal{M} \rrbracket \implies C1 \cap C2\ face_of\ C1$

obtains \mathcal{T} where *simplicial_complex* \mathcal{T}

$\bigwedge K. K \in \mathcal{T} \implies diameter\ K < e$

$\bigcup \mathcal{T} = \bigcup \mathcal{M}$

$\bigwedge C. C \in \mathcal{M} \implies \exists F. finite\ F \wedge F \subseteq \mathcal{T} \wedge C = \bigcup F$

$\bigwedge K. K \in \mathcal{T} \implies \exists C. C \in \mathcal{M} \wedge K \subseteq C$

9.24.20 Some results on cell division with full-dimensional cells only

proposition *fine_triangular_subdivision_of_cell_complex:*

assumes $0 < e$ *finite* \mathcal{M}

and *poly*: $\bigwedge C. C \in \mathcal{M} \implies \text{polytope } C$

and *aff*: $\bigwedge C. C \in \mathcal{M} \implies \text{aff_dim } C = d$

and *face*: $\bigwedge C1\ C2. \llbracket C1 \in \mathcal{M}; C2 \in \mathcal{M} \rrbracket \implies C1 \cap C2 \text{ face_of } C1$

obtains \mathcal{T} **where** *triangulation* $\mathcal{T} \bigwedge k. k \in \mathcal{T} \implies \text{diameter } k < e$

$\bigwedge k. k \in \mathcal{T} \implies \text{aff_dim } k = d \cup \mathcal{T} = \cup \mathcal{M}$

$\bigwedge C. C \in \mathcal{M} \implies \exists f. \text{finite } f \wedge f \subseteq \mathcal{T} \wedge C = \cup f$

$\bigwedge k. k \in \mathcal{T} \implies \exists C. C \in \mathcal{M} \wedge k \subseteq C$

9.25 Finitely generated cone is polyhedral, and hence closed

proposition *polyhedron_convex_cone_hull:*

fixes $S :: 'a::\text{euclidean_space set}$

assumes *finite* S

shows *polyhedron*(*convex_cone hull* S)

end

9.26 Absolute Retracts, Absolute Neighbourhood Retracts and Euclidean Neighbourhood Retracts

theory *Retracts*

imports

Brouwer_Fixpoint

Continuous_Extension

begindefinition *AR* :: $'a::\text{topological_space set} \Rightarrow \text{bool}$ **where**

AR $S \equiv \forall U. \forall S'::('a * \text{real}) \text{ set.}$

$S \text{ homeomorphic } S' \wedge \text{closedin } (\text{top_of_set } U) S' \longrightarrow S' \text{ retract_of } U$

definition *ANR* :: $'a::\text{topological_space set} \Rightarrow \text{bool}$ **where**

ANR $S \equiv \forall U. \forall S'::('a * \text{real}) \text{ set.}$

$S \text{ homeomorphic } S' \wedge \text{closedin } (\text{top_of_set } U) S'$

$\longrightarrow (\exists T. \text{openin } (\text{top_of_set } U) T \wedge S' \text{ retract_of } T)$

definition *ENR* :: $'a::\text{topological_space set} \Rightarrow \text{bool}$ **where**

ENR $S \equiv \exists U. \text{open } U \wedge S \text{ retract_of } U$

corollary *ANR_imp_absolute_neighbourhood_retract*:
fixes $S :: 'a::\text{euclidean_space set}$ **and** $S' :: 'b::\text{euclidean_space set}$
assumes *ANR S S homeomorphic S'*
and $\text{clo}: \text{closedin } (\text{top_of_set } U) S'$
obtains V **where** $\text{openin } (\text{top_of_set } U) V$ S' *retract_of V*

corollary *ANR_imp_absolute_neighbourhood_retract_UNIV*:
fixes $S :: 'a::\text{euclidean_space set}$ **and** $S' :: 'b::\text{euclidean_space set}$
assumes *ANR S* **and** $\text{hom}: S$ *homeomorphic S'* **and** $\text{clo}: \text{closed } S'$
obtains V **where** $\text{open } V$ S' *retract_of V*

corollary *neighbourhood_extension_into_ANR*:
fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{contf}: \text{continuous_on } S f$ **and** $\text{fim}: f \in S \rightarrow T$ **and** *ANR T closed S*
obtains $V g$ **where** $S \subseteq V$ $\text{open } V$ $\text{continuous_on } V g$
 $g \in V \rightarrow T \wedge x. x \in S \implies g x = f x$

9.26.1 Analogous properties of ENRs

corollary *ENR_imp_absolute_neighbourhood_retract_UNIV*:
fixes $S :: 'a::\text{euclidean_space set}$ **and** $S' :: 'b::\text{euclidean_space set}$
assumes *ENR S S homeomorphic S'*
obtains T' **where** $\text{open } T'$ S' *retract_of T'*

corollary *AR_closed_Un*:
fixes $S :: 'a::\text{euclidean_space set}$
shows $[\text{closed } S; \text{closed } T; \text{AR } S; \text{AR } T; \text{AR } (S \cap T)] \implies \text{AR } (S \cup T)$

corollary *ANR_closed_Un*:
fixes $S :: 'a::\text{euclidean_space set}$
shows $[\text{closed } S; \text{closed } T; \text{ANR } S; \text{ANR } T; \text{ANR } (S \cap T)] \implies \text{ANR } (S \cup T)$

9.26.2 More advanced properties of ANRs and ENRs

9.26.3 Original ANR material, now for ENRs

9.26.4 Finally, spheres are ANRs and ENRs

9.26.5 Spheres are connected, etc

9.26.6 Borsuk homotopy extension theorem

theorem *Borsuk_homotopy_extension_homotopic*:
fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $cloTS: closedin (top_of_set T) S$
and $anr: (ANR S \wedge ANR T) \vee ANR U$
and $contf: continuous_on T f$
and $f \in T \rightarrow U$
and $homotopic_with_canon (\lambda x. True) S U f g$
obtains g' **where** $homotopic_with_canon (\lambda x. True) T U f g'$
 $continuous_on T g' image g' T \subseteq U$
 $\bigwedge x. x \in S \implies g' x = g x$

9.26.7 More extension theorems

9.26.8 The complement of a set and path-connectedness

theorem *connected_complement_homeomorphic_convex_compact*:
fixes $S :: 'a::euclidean_space set$ **and** $T :: 'b::euclidean_space set$
assumes $hom: S homeomorphic T$ **and** $T: convex T compact T$ **and** $2: 2 \leq DIM('a)$
shows $connected(- S)$

corollary *path_connected_complement_homeomorphic_convex_compact*:
fixes $S :: 'a::euclidean_space set$ **and** $T :: 'b::euclidean_space set$
assumes $hom: S homeomorphic T$ $convex T compact T$ $2 \leq DIM('a)$
shows $path_connected(- S)$

end

9.27 Extending Continuous Maps, Invariance of Domain, etc

theory *Further_Topology*
imports *Weierstrass_Theorems Polytope Complex_Transcendental Equivalence_Lebesgue_Henstock_Integration Retracts*
begin

9.27.1 A map from a sphere to a higher dimensional sphere is nullhomotopic

proposition *inessential_spheremap_lowdim_gen:*

fixes $f :: 'M::euclidean_space \Rightarrow 'a::euclidean_space$

assumes $convex\ S\ bounded\ S\ convex\ T\ bounded\ T$

and $affST: aff_dim\ S < aff_dim\ T$

and $contf: continuous_on\ (rel_frontier\ S)\ f$

and $fim: f \in (rel_frontier\ S) \rightarrow rel_frontier\ T$

obtains c **where** $homotopic_with_canon\ (\lambda z. True)\ (rel_frontier\ S)\ (rel_frontier\ T)\ f\ (\lambda x. c)$

9.27.2 Some technical lemmas about extending maps from cell complexes

theorem *extend_map_cell_complex_to_sphere:*

assumes $finite\ \mathcal{F}$ **and** $S: S \subseteq \bigcup \mathcal{F}$ $closed\ S$ **and** $T: convex\ T\ bounded\ T$

and $poly: \bigwedge X. X \in \mathcal{F} \Rightarrow polytope\ X$

and $aff: \bigwedge X. X \in \mathcal{F} \Rightarrow aff_dim\ X < aff_dim\ T$

and $face: \bigwedge X\ Y. \llbracket X \in \mathcal{F}; Y \in \mathcal{F} \rrbracket \Rightarrow (X \cap Y)\ face_of\ X$

and $contf: continuous_on\ S\ f$ **and** $fim: f \in S \rightarrow rel_frontier\ T$

obtains g **where** $continuous_on\ (\bigcup \mathcal{F})\ g$

$g \in (\bigcup \mathcal{F}) \rightarrow rel_frontier\ T \ \bigwedge x. x \in S \Rightarrow g\ x = f\ x$

theorem *extend_map_cell_complex_to_sphere_cofinite:*

assumes $finite\ \mathcal{F}$ **and** $S: S \subseteq \bigcup \mathcal{F}$ $closed\ S$ **and** $T: convex\ T\ bounded\ T$

and $poly: \bigwedge X. X \in \mathcal{F} \Rightarrow polytope\ X$

and $aff: \bigwedge X. X \in \mathcal{F} \Rightarrow aff_dim\ X \leq aff_dim\ T$

and $face: \bigwedge X\ Y. \llbracket X \in \mathcal{F}; Y \in \mathcal{F} \rrbracket \Rightarrow (X \cap Y)\ face_of\ X$

and $contf: continuous_on\ S\ f$ **and** $fim: f \in S \rightarrow rel_frontier\ T$

obtains $C\ g$ **where** $finite\ C\ disjoint\ C\ S\ continuous_on\ (\bigcup \mathcal{F} - C)\ g$

$g \in (\bigcup \mathcal{F} - C) \rightarrow rel_frontier\ T \ \bigwedge x. x \in S \Rightarrow g\ x = f\ x$

9.27.3 Special cases and corollaries involving spheres

proposition *extend_map_affine_to_sphere_cofinite_simple:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$

assumes $compact\ S\ convex\ U\ bounded\ U$

and $aff: aff_dim\ T \leq aff_dim\ U$

and $S \subseteq T$ **and** $contf: continuous_on\ S\ f$

and $fim: f \in S \rightarrow rel_frontier\ U$

obtains $K g$ **where** $finite\ K\ K \subseteq T\ disjnt\ K\ S\ continuous_on\ (T - K)\ g$
 $g \in (T - K) \rightarrow rel_frontier\ U$
 $\bigwedge x. x \in S \implies g\ x = f\ x$

9.27.4 Extending maps to spheres

proposition *extend_map_affine_to_sphere_cofinite_gen:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $SUT: compact\ S\ convex\ U\ bounded\ U\ affine\ T\ S \subseteq T$
and $aff: aff_dim\ T \leq aff_dim\ U$
and $contf: continuous_on\ S\ f$
and $fm: f \in S \rightarrow rel_frontier\ U$
and $dis: \bigwedge C. \llbracket C \in components(T - S); bounded\ C \rrbracket \implies C \cap L \neq \{\}$
obtains $K g$ **where** $finite\ K\ K \subseteq L\ K \subseteq T\ disjnt\ K\ S\ continuous_on\ (T - K)$
 g
 $g \in (T - K) \rightarrow rel_frontier\ U$
 $\bigwedge x. x \in S \implies g\ x = f\ x$

corollary *extend_map_affine_to_sphere_cofinite:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $SUT: compact\ S\ affine\ T\ S \subseteq T$
and $aff: aff_dim\ T \leq DIM('b)$ **and** $0 \leq r$
and $contf: continuous_on\ S\ f$
and $fm: f \in S \rightarrow sphere\ a\ r$
and $dis: \bigwedge C. \llbracket C \in components(T - S); bounded\ C \rrbracket \implies C \cap L \neq \{\}$
obtains $K g$ **where** $finite\ K\ K \subseteq L\ K \subseteq T\ disjnt\ K\ S\ continuous_on\ (T - K)$
 g
 $g \in (T - K) \rightarrow sphere\ a\ r \bigwedge x. x \in S \implies g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_cofinite:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $DIM('a) \leq DIM('b)$ **and** $0 \leq r$
and $compact\ S$
and $continuous_on\ S\ f$
and $f \in S \rightarrow sphere\ a\ r$
and $\bigwedge C. \llbracket C \in components(-\ S); bounded\ C \rrbracket \implies C \cap L \neq \{\}$
obtains $K g$ **where** $finite\ K\ K \subseteq L\ disjnt\ K\ S\ continuous_on\ (-\ K)\ g$
 $g \in (-\ K) \rightarrow sphere\ a\ r \bigwedge x. x \in S \implies g\ x = f\ x$

corollary *extend_map_UNIV_to_sphere_no_bounded_component:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $aff: DIM('a) \leq DIM('b)$ **and** $0 \leq r$
and $SUT: compact\ S$
and $contf: continuous_on\ S\ f$

and *fm*: $f \in S \rightarrow \text{sphere } a \ r$
and *dis*: $\bigwedge C. C \in \text{components}(- S) \implies \neg \text{bounded } C$
obtains *g* **where** *continuous_on UNIV* $g \ g \in \text{UNIV} \rightarrow \text{sphere } a \ r \ \bigwedge x. x \in S$
 $\implies g \ x = f \ x$

theorem *Borsuk_separation_theorem_gen*:

fixes *S* :: 'a::euclidean_space set

assumes *compact S*

shows $(\forall c \in \text{components}(- S). \neg \text{bounded } c) \longleftrightarrow$

$(\forall f. \text{continuous_on } S \ f \wedge f \in S \rightarrow \text{sphere } (0::'a) \ 1$

$\longrightarrow (\exists c. \text{homotopic_with_canon } (\lambda x. \text{True}) \ S \ (\text{sphere } 0 \ 1) \ f \ (\lambda x.$

c)))

(**is** ?lhs = ?rhs)

corollary *Borsuk_separation_theorem*:

fixes *S* :: 'a::euclidean_space set

assumes *compact S* **and** *2*: $2 \leq \text{DIM}('a)$

shows $\text{connected}(- S) \longleftrightarrow$

$(\forall f. \text{continuous_on } S \ f \wedge f \in S \rightarrow \text{sphere } (0::'a) \ 1$

$\longrightarrow (\exists c. \text{homotopic_with_canon } (\lambda x. \text{True}) \ S \ (\text{sphere } 0 \ 1) \ f \ (\lambda x.$

c)))

(**is** ?lhs = ?rhs)

proposition *Jordan_Brouwer_separation*:

fixes *S* :: 'a::euclidean_space set **and** *a*::'a

assumes *hom*: *S* homeomorphic sphere *a* *r* **and** $0 < r$

shows $\neg \text{connected}(- S)$

proposition *Jordan_Brouwer_frontier*:

fixes *S* :: 'a::euclidean_space set **and** *a*::'a

assumes *S*: *S* homeomorphic sphere *a* *r* **and** *T*: $T \in \text{components}(- S)$ **and** *2*:
 $2 \leq \text{DIM}('a)$

shows *frontier* $T = S$

proposition *Jordan_Brouwer_nonseparation*:

fixes *S* :: 'a::euclidean_space set **and** *a*::'a

assumes *S*: *S* homeomorphic sphere *a* *r* **and** $T \subset S$ **and** *2*: $2 \leq \text{DIM}('a)$

shows $\text{connected}(- T)$

9.27.5 Invariance of domain and corollaries

theorem *invariance_of_domain*:

fixes *f* :: 'a \Rightarrow 'a::euclidean_space

assumes *continuous_on S* *f* *open S* *inj_on f S*

shows $open(f \text{ ` } S)$

corollary *invariance_of_domain_subspaces:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $ope: openin (top_of_set U) S$
and $subspace U \text{ } subspace V$ **and** $VU: dim V \leq dim U$
and $contf: continuous_on S f$ **and** $fm: f \in S \rightarrow V$
and $injf: inj_on f S$
shows $openin (top_of_set V) (f \text{ ` } S)$

corollary *invariance_of_dimension_subspaces:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $ope: openin (top_of_set U) S$
and $subspace U \text{ } subspace V$
and $contf: continuous_on S f$ **and** $fm: f \in S \rightarrow V$
and $injf: inj_on f S$ **and** $S \neq \{\}$
shows $dim U \leq dim V$

corollary *invariance_of_domain_affine_sets:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $ope: openin (top_of_set U) S$
and $aff: affine U \text{ } affine V$ $aff_dim V \leq aff_dim U$
and $contf: continuous_on S f$ **and** $fm: f \in S \rightarrow V$
and $injf: inj_on f S$
shows $openin (top_of_set V) (f \text{ ` } S)$

corollary *invariance_of_dimension_affine_sets:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $ope: openin (top_of_set U) S$
and $aff: affine U \text{ } affine V$
and $contf: continuous_on S f$ **and** $fm: f \in S \rightarrow V$
and $injf: inj_on f S$ **and** $S \neq \{\}$
shows $aff_dim U \leq aff_dim V$

corollary *invariance_of_dimension:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $contf: continuous_on S f$ **and** $open S$
and $injf: inj_on f S$ **and** $S \neq \{\}$
shows $DIM('a) \leq DIM('b)$

corollary *continuous_injective_image_subspace_dim_le:*

fixes $f :: 'a::euclidean_space \Rightarrow 'b::euclidean_space$
assumes $subspace S \text{ } subspace T$
and $contf: continuous_on S f$ **and** $fm: f \in S \rightarrow T$
and $injf: inj_on f S$
shows $dim S \leq dim T$

corollary *invariance_of_domain_homeomorphic:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $\text{open } S \text{ continuous_on } S f \text{ DIM}('b) \leq \text{DIM}('a) \text{ inj_on } f S$
shows $S \text{ homeomorphic } (f \text{ ` } S)$

proposition *homeomorphic_interiors:*

fixes $S :: 'a::\text{euclidean_space set}$ **and** $T :: 'b::\text{euclidean_space set}$
assumes $S \text{ homeomorphic } T \text{ interior } S = \{\} \longleftrightarrow \text{interior } T = \{\}$
shows $(\text{interior } S) \text{ homeomorphic } (\text{interior } T)$

proposition *uniformly_continuous_homeomorphism_UNIV_trivial:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'a$
assumes $\text{contf: uniformly_continuous_on } S f$ **and** $\text{hom: homeomorphism } S$
 $\text{UNIV } f g$
shows $S = \text{UNIV}$

9.27.6 Formulation of loop homotopy in terms of maps out of type complex

proposition *simply_connected_eq_homotopic_circlemaps:*

fixes $S :: 'a::\text{real_normed_vector set}$
shows $\text{simply_connected } S \longleftrightarrow$
 $(\forall f g :: \text{complex} \Rightarrow 'a.$
 $\text{continuous_on } (\text{sphere } 0 \ 1) f \wedge f \in (\text{sphere } 0 \ 1) \rightarrow S \wedge$
 $\text{continuous_on } (\text{sphere } 0 \ 1) g \wedge g \in (\text{sphere } 0 \ 1) \rightarrow S$
 $\longrightarrow \text{homotopic_with_canon } (\lambda h. \text{True}) (\text{sphere } 0 \ 1) S f g)$

proposition *simply_connected_eq_contractible_circlemap:*

fixes $S :: 'a::\text{real_normed_vector set}$
shows $\text{simply_connected } S \longleftrightarrow$
 $\text{path_connected } S \wedge$
 $(\forall f :: \text{complex} \Rightarrow 'a.$
 $\text{continuous_on } (\text{sphere } 0 \ 1) f \wedge f \text{ ` } (\text{sphere } 0 \ 1) \subseteq S$
 $\longrightarrow (\exists a. \text{homotopic_with_canon } (\lambda h. \text{True}) (\text{sphere } 0 \ 1) S f (\lambda x. a)))$

corollary *homotopy_eqv_simple_connectedness:*

fixes $S :: 'a::\text{real_normed_vector set}$ **and** $T :: 'b::\text{real_normed_vector set}$
shows $S \text{ homotopy_eqv } T \implies \text{simply_connected } S \longleftrightarrow \text{simply_connected } T$

9.27.7 Homeomorphism of simple closed curves to circles

proposition *homeomorphic_simple_path_image_circle:*

fixes $a :: \text{complex}$ and $\gamma :: \text{real} \Rightarrow 'a::t2_space$

assumes *simple_path* γ and *loop*: $\text{pathfinish } \gamma = \text{pathstart } \gamma$ and $0 < r$

shows $(\text{path_image } \gamma)$ *homeomorphic sphere* a r

9.27.8 Dimension-based conditions for various homeomorphisms

9.27.9 more invariance of domain

proposition *invariance_of_domain_sphere_affine_set_gen:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$

assumes *contf*: *continuous_on* S f and *injf*: *inj_on* f S and *fim*: $f \in S \rightarrow T$

and U : *bounded* U *convex* U

and *affine* T and *affTU*: $\text{aff_dim } T < \text{aff_dim } U$

and *ope*: *openin* $(\text{top_of_set } (\text{rel_frontier } U))$ S

shows *openin* $(\text{top_of_set } T)$ $(f \text{ ` } S)$

proposition *simply_connected_punctured_convex:*

fixes $a :: 'a::\text{euclidean_space}$

assumes *convex* S and \exists : $\exists \leq \text{aff_dim } S$

shows *simply_connected* $(S - \{a\})$

corollary *simply_connected_punctured_universe:*

fixes $a :: 'a::\text{euclidean_space}$

assumes $\exists \leq \text{DIM}('a)$

shows *simply_connected* $(- \{a\})$

9.27.10 The power, squaring and exponential functions as covering maps

proposition *covering_space_power_punctured_plane:*

assumes $0 < n$

shows *covering_space* $(- \{0\})$ $(\lambda z::\text{complex. } z^n)$ $(- \{0\})$

corollary *covering_space_square_punctured_plane:*

covering_space $(- \{0\})$ $(\lambda z::\text{complex. } z^2)$ $(- \{0\})$

proposition *covering_space_exp_punctured_plane:*

covering_space UNIV $(\lambda z::\text{complex. } \exp z)$ $(- \{0\})$

9.27.11 Hence the Borsukian results about mappings into circles

corollary *inessential_imp_continuous_logarithm_circle:*

fixes $f :: 'a::\text{real_normed_vector} \Rightarrow \text{complex}$

assumes $\text{homotopic_with_canon } (\lambda h. \text{True}) S (\text{sphere } 0 \ 1) f (\lambda t. a)$

obtains g **where** $\text{continuous_on } S \ g$ **and** $\bigwedge x. x \in S \implies f \ x = \exp(g \ x)$

proposition *homotopic_with_sphere_times:*

fixes $f :: 'a::\text{real_normed_vector} \Rightarrow \text{complex}$

assumes $\text{homotopic_with_canon } (\lambda x. \text{True}) S (\text{sphere } 0 \ 1) f \ g$ **and** $\text{conth: continuous_on } S \ h$

and $\text{hin: } \bigwedge x. x \in S \implies h \ x \in \text{sphere } 0 \ 1$

shows $\text{homotopic_with_canon } (\lambda x. \text{True}) S (\text{sphere } 0 \ 1) (\lambda x. f \ x * h \ x) (\lambda x. g \ x * h \ x)$

proposition *homotopic_circlemaps_divide:*

fixes $f :: 'a::\text{real_normed_vector} \Rightarrow \text{complex}$

shows $\text{homotopic_with_canon } (\lambda x. \text{True}) S (\text{sphere } 0 \ 1) f \ g \longleftrightarrow$

$\text{continuous_on } S \ f \wedge f \in S \rightarrow \text{sphere } 0 \ 1 \wedge$

$\text{continuous_on } S \ g \wedge g \in S \rightarrow \text{sphere } 0 \ 1 \wedge$

$(\exists c. \text{homotopic_with_canon } (\lambda x. \text{True}) S (\text{sphere } 0 \ 1) (\lambda x. f \ x / g \ x))$
($\lambda x. c$)

9.27.12 Upper and lower hemicontinuous functions

proposition *upper_lower_hemicontinuous_explicit:*

fixes $T :: ('b::\{\text{real_normed_vector}, \text{heine_borel}\}) \text{ set}$

assumes $fST: \bigwedge x. x \in S \implies f \ x \subseteq T$

and $\text{ope: } \bigwedge U. \text{openin } (\text{top_of_set } T) \ U$

$\implies \text{openin } (\text{top_of_set } S) \ \{x \in S. f \ x \subseteq U\}$

and $\text{clo: } \bigwedge U. \text{closedin } (\text{top_of_set } T) \ U$

$\implies \text{closedin } (\text{top_of_set } S) \ \{x \in S. f \ x \subseteq U\}$

and $x \in S \ 0 < e$ **and** $\text{bofx: bounded}(f \ x)$ **and** $\text{fx_ne: } f \ x \neq \{\}$

obtains d **where** $0 < d$

$\bigwedge x'. \llbracket x' \in S; \text{dist } x \ x' < d \rrbracket$

$\implies (\forall y \in f \ x. \exists y'. y' \in f \ x' \wedge \text{dist } y \ y' < e) \wedge$

$(\forall y' \in f \ x'. \exists y. y \in f \ x \wedge \text{dist } y' \ y < e)$

- 9.27.13 Complex logs exist on various "well-behaved" sets
- 9.27.14 Another simple case where sphere maps are nullhomotopic
- 9.27.15 Holomorphic logarithms and square roots

9.27.16 The "Borsukian" property of sets

definition *Borsukian* where

$$\begin{aligned} \text{Borsukian } S \equiv & \\ & \forall f. \text{continuous_on } S \ f \wedge f \in S \rightarrow (- \{0::\text{complex}\}) \\ & \rightarrow (\exists a. \text{homotopic_with_canon } (\lambda h. \text{True}) \ S \ (- \{0\}) \ f \ (\lambda x. a)) \end{aligned}$$

proposition *Borsukian_sphere*:

fixes $a :: 'a::\text{euclidean_space}$
shows $3 \leq \text{DIM}('a) \implies \text{Borsukian } (\text{sphere } a \ r)$

proposition *Borsukian_open_Un*:

fixes $S :: 'a::\text{real_normed_vector_set}$
assumes $\text{opeS}: \text{openin } (\text{top_of_set } (S \cup T)) \ S$
and $\text{opeT}: \text{openin } (\text{top_of_set } (S \cup T)) \ T$
and $\text{BS}: \text{Borsukian } S$ **and** $\text{BT}: \text{Borsukian } T$ **and** $\text{ST}: \text{connected}(S \cap T)$
shows $\text{Borsukian}(S \cup T)$

proposition *closed_irreducible_separator*:

fixes $a :: 'a::\text{real_normed_vector}$
assumes $\text{closed } S$ **and** $\text{ab}: \neg \text{connected_component } (- S) \ a \ b$
obtains T **where** $T \subseteq S$ $\text{closed } T$ $T \neq \{\}$ $\neg \text{connected_component } (- T) \ a \ b$
 $\wedge U. U \subset T \implies \text{connected_component } (- U) \ a \ b$

9.27.17 Unicoherence (closed)

definition *unicoherent* where

$$\begin{aligned} \text{unicoherent } U \equiv & \\ & \forall S \ T. \text{connected } S \wedge \text{connected } T \wedge S \cup T = U \wedge \\ & \text{closedin } (\text{top_of_set } U) \ S \wedge \text{closedin } (\text{top_of_set } U) \ T \\ & \rightarrow \text{connected } (S \cap T) \end{aligned}$$

proposition *homeomorphic_unicoherent*:

assumes $\text{ST}: S \text{ homeomorphic } T$ **and** $S: \text{unicoherent } S$
shows $\text{unicoherent } T$

corollary *contractible_imp_unicoherent*:

fixes $U :: 'a::\text{euclidean_space set}$
assumes *contractible* U **shows** *unicoherent* U

corollary *convex_imp_unicoherent*:

fixes $U :: 'a::\text{euclidean_space set}$
assumes *convex* U **shows** *unicoherent* U

corollary *unicoherent_UNIV*: *unicoherent* ($UNIV :: 'a :: \text{euclidean_space set}$)

9.27.18 Several common variants of unicoherence

9.27.19 Some separation results

proposition *separation_by_component_open*:

fixes $S :: 'a :: \text{euclidean_space set}$
assumes *open* S **and** *non*: $\neg \text{connected}(- S)$
obtains C **where** $C \in \text{components } S \neg \text{connected}(- C)$

proposition *inessential_eq_extensible*:

fixes $f :: 'a::\text{euclidean_space} \Rightarrow \text{complex}$
assumes *closed* S
shows $(\exists a. \text{homotopic_with_canon } (\lambda h. \text{True}) S (-\{0\}) f (\lambda t. a)) \longleftrightarrow$
 $(\exists g. \text{continuous_on } UNIV g \wedge (\forall x \in S. g x = f x) \wedge (\forall x. g x \neq 0))$
(is *?lhs = ?rhs***)**

proposition *Janiszewski_dual*:

fixes $S :: \text{complex set}$
assumes *compact* S *compact* T *connected* S *connected* T *connected* $(- (S \cup T))$
shows *connected* $(S \cap T)$

end

9.28 The Jordan Curve Theorem and Applications

theory *Jordan_Curve*

imports *Arcwise_Connected Further_Topology*

begin

9.28.1 Janiszewski's theorem

theorem *Janiszewski*:

fixes $a\ b :: \text{complex}$
assumes $\text{compact } S\ \text{closed } T$ **and** $\text{con}ST: \text{connected } (S \cap T)$
and $\text{cc}S: \text{connected_component } (-\ S)\ a\ b$ **and** $\text{cc}T: \text{connected_component } (-\ T)\ a\ b$
shows $\text{connected_component } (-\ (S \cup T))\ a\ b$

9.28.2 The Jordan Curve theorem

corollary *Jordan_inside_outside:*

fixes $c :: \text{real} \Rightarrow \text{complex}$
assumes $\text{simple_path } c\ \text{pathfinish } c = \text{pathstart } c$
shows $\text{inside}(\text{path_image } c) \neq \{\}$ \wedge
 $\text{open}(\text{inside}(\text{path_image } c)) \wedge$
 $\text{connected}(\text{inside}(\text{path_image } c)) \wedge$
 $\text{outside}(\text{path_image } c) \neq \{\}$ \wedge
 $\text{open}(\text{outside}(\text{path_image } c)) \wedge$
 $\text{connected}(\text{outside}(\text{path_image } c)) \wedge$
 $\text{bounded}(\text{inside}(\text{path_image } c)) \wedge$
 $\neg \text{bounded}(\text{outside}(\text{path_image } c)) \wedge$
 $\text{inside}(\text{path_image } c) \cap \text{outside}(\text{path_image } c) = \{\}$ \wedge
 $\text{inside}(\text{path_image } c) \cup \text{outside}(\text{path_image } c) =$
 $-\ \text{path_image } c \wedge$
 $\text{frontier}(\text{inside}(\text{path_image } c)) = \text{path_image } c \wedge$
 $\text{frontier}(\text{outside}(\text{path_image } c)) = \text{path_image } c$

theorem *split_inside_simple_closed_curve:*

fixes $c :: \text{real} \Rightarrow \text{complex}$
assumes $\text{simple_path } c1$ **and** $c1: \text{pathstart } c1 = a\ \text{pathfinish } c1 = b$
and $\text{simple_path } c2$ **and** $c2: \text{pathstart } c2 = a\ \text{pathfinish } c2 = b$
and $\text{simple_path } c$ **and** $c: \text{pathstart } c = a\ \text{pathfinish } c = b$
and $a \neq b$
and $c1c2: \text{path_image } c1 \cap \text{path_image } c2 = \{a,b\}$
and $c1c: \text{path_image } c1 \cap \text{path_image } c = \{a,b\}$
and $c2c: \text{path_image } c2 \cap \text{path_image } c = \{a,b\}$
and $\text{ne_12}: \text{path_image } c \cap \text{inside}(\text{path_image } c1 \cup \text{path_image } c2) \neq \{\}$
obtains $\text{inside}(\text{path_image } c1 \cup \text{path_image } c) \cap \text{inside}(\text{path_image } c2 \cup$
 $\text{path_image } c) = \{\}$
 $\text{inside}(\text{path_image } c1 \cup \text{path_image } c) \cup \text{inside}(\text{path_image } c2 \cup$
 $\text{path_image } c) \cup$
 $(\text{path_image } c - \{a,b\}) = \text{inside}(\text{path_image } c1 \cup \text{path_image } c2)$

end

9.29 Polynomial Functions: Extremal Behaviour and Root Counts

```
theory Poly_Roots
imports Complex_Main
begin
```

9.29.1 Basics about polynomial functions: extremal behaviour and root counts

```
proposition polyfun_extremal_lemma:
  fixes c :: nat ⇒ 'a::real_normed_div_algebra
  assumes e > 0
  shows ∃ M. ∀ z. M ≤ norm z ⟶ norm(∑ i≤n. c i * zi) ≤ e * norm(z) ^
Suc n
```

```
proposition polyfun_extremal:
  fixes c :: nat ⇒ 'a::real_normed_div_algebra
  assumes ∃ k. k ≠ 0 ∧ k ≤ n ∧ c k ≠ 0
  shows eventually (λz. norm(∑ i≤n. c i * zi) ≥ B) at_infinity
```

```
proposition polyfun_rootbound:
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
  assumes ∃ k. k ≤ n ∧ c k ≠ 0
  shows finite {z. (∑ i≤n. c i * zi) = 0} ∧ card {z. (∑ i≤n. c i * zi) = 0}
≤ n
```

```
corollary
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
  assumes ∃ k. k ≤ n ∧ c k ≠ 0
  shows polyfun_rootbound_finite: finite {z. (∑ i≤n. c i * zi) = 0}
  and polyfun_rootbound_card: card {z. (∑ i≤n. c i * zi) = 0} ≤ n
```

```
proposition polyfun_finite_roots:
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
  shows finite {z. (∑ i≤n. c i * zi) = 0} ⟷ (∃ k. k ≤ n ∧ c k ≠ 0)
```

```
theorem polyfun_eq_const:
  fixes c :: nat ⇒ 'a::{comm_ring,real_normed_div_algebra}
  shows (∀ z. (∑ i≤n. c i * zi) = k) ⟷ c 0 = k ∧ (∀ k. k ≠ 0 ∧ k ≤ n ⟶
c k = 0)
```

```
end
```

9.30 Generalised Binomial Theorem

```
theory Generalised_Binomial_Theorem
```

```

imports
  Complex_Main
  Complex_Transcendental
  Summation_Tests
begin

theorem gen_binomial_complex:
  fixes  $z :: \text{complex}$ 
  assumes  $\text{norm } z < 1$ 
  shows  $(\lambda n. (a \text{ choose } n) * z^n) \text{ sums } (1 + z) \text{ powr } a$ 

end

```

9.31 Vitali Covering Theorem and an Application to Negligibility

```

theory Vitali_Covering_Theorem
imports
  HOL-Combinatorics.Permutations
  Equivalence_Lebesgue_Henstock_Integration
begin

```

9.31.1 Vitali covering theorem

```

theorem Vitali_covering_theorem_cballs:
  fixes  $a :: 'a \Rightarrow 'n::\text{euclidean\_space}$ 
  assumes  $r: \bigwedge i. i \in K \implies 0 < r\ i$ 
  and  $S: \bigwedge x\ d. \llbracket x \in S; 0 < d \rrbracket$ 
   $\implies \exists i. i \in K \wedge x \in \text{cball } (a\ i) (r\ i) \wedge r\ i < d$ 
  obtains  $C$  where countable  $C\ C \subseteq K$ 
  pairwise  $(\lambda i\ j. \text{disjnt } (\text{cball } (a\ i) (r\ i)) (\text{cball } (a\ j) (r\ j)))\ C$ 
  negligible  $(S - (\bigcup i \in C. \text{cball } (a\ i) (r\ i)))$ 

```

```

theorem Vitali_covering_theorem_balls:
  fixes  $a :: 'a \Rightarrow 'b::\text{euclidean\_space}$ 
  assumes  $S: \bigwedge x\ d. \llbracket x \in S; 0 < d \rrbracket \implies \exists i. i \in K \wedge x \in \text{ball } (a\ i) (r\ i) \wedge r\ i < d$ 
  obtains  $C$  where countable  $C\ C \subseteq K$ 
  pairwise  $(\lambda i\ j. \text{disjnt } (\text{ball } (a\ i) (r\ i)) (\text{ball } (a\ j) (r\ j)))\ C$ 
  negligible  $(S - (\bigcup i \in C. \text{ball } (a\ i) (r\ i)))$ 

```

```

proposition negligible_eq_zero_density:
  negligible  $S \iff$ 

```

$$(\forall x \in S. \forall r > 0. \forall e > 0. \exists d. 0 < d \wedge d \leq r \wedge$$

$$(\exists U. S \cap \text{ball } x \ d \subseteq U \wedge U \in \text{lmeasurable} \wedge \text{measure lebesgue } U$$

$$< e * \text{measure lebesgue } (\text{ball } x \ d)))$$

end

9.32 Change of Variables Theorems

theory *Change_Of_Vars*
imports *Vitali_Covering_Theorem Determinants*

begin

9.32.1 Measurable Shear and Stretch

proposition

fixes $a :: \text{real}^n$
assumes $m \neq n$ **and** $ab_ne: \text{cbox } a \ b \neq \{\}$ **and** $an: 0 \leq a\$n$
shows $\text{measurable_shear_interval}: (\lambda x. \chi \ i. \text{if } i = m \text{ then } x\$m + x\$n \text{ else } x\$i)$
 $'(\text{cbox } a \ b) \in \text{lmeasurable}$
(is $?f \ ' _ \in _)$
and $\text{measure_shear_interval}: \text{measure lebesgue } ((\lambda x. \chi \ i. \text{if } i = m \text{ then } x\$m +$
 $x\$n \text{ else } x\$i) \ ' \text{cbox } a \ b)$
 $= \text{measure lebesgue } (\text{cbox } a \ b)$ **(is** $?Q)$

proposition

fixes $S :: (\text{real}^n)$ *set*
assumes $S \in \text{lmeasurable}$
shows $\text{measurable_stretch}: ((\lambda x. \chi \ k. m \ k * x\$k) \ ' S) \in \text{lmeasurable}$ **(is** $?f \ ' S$
 $\in _)$
and $\text{measure_stretch}: \text{measure lebesgue } ((\lambda x. \chi \ k. m \ k * x\$k) \ ' S) = |\text{prod } m$
 $\text{UNIV}| * \text{measure lebesgue } S$
(is $?MEQ)$

proposition

fixes $f :: \text{real}^n :: \{\text{finite, wellorder}\} \Rightarrow \text{real}^n :: _$
assumes $\text{linear } f \ S \in \text{lmeasurable}$
shows $\text{measurable_linear_image}: (f \ ' S) \in \text{lmeasurable}$
and $\text{measure_linear_image}: \text{measure lebesgue } (f \ ' S) = |\text{det } (\text{matrix } f)| *$
 $\text{measure lebesgue } S$ **(is** $?Q \ f \ S)$

proposition *measure_semicontinuous_with_hausdist_explicit:***assumes** *bounded* S **and** *neg: negligible*(*frontier* S) **and** $e > 0$ **obtains** d **where** $d > 0$

$$\bigwedge T. \llbracket T \in \text{lmeasurable}; \bigwedge y. y \in T \implies \exists x. x \in S \wedge \text{dist } x \ y < d \rrbracket$$

$$\implies \text{measure lebesgue } T < \text{measure lebesgue } S + e$$

proposition

fixes $f :: \text{real}^n :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n :: _$
assumes $S: S \in \text{lmeasurable}$
and deriv: $\bigwedge x. x \in S \implies (f \text{ has_derivative } f' x) \text{ (at } x \text{ within } S)$
and int: $(\lambda x. |\det (\text{matrix } (f' x))|) \text{ integrable_on } S$
and bounded: $\bigwedge x. x \in S \implies |\det (\text{matrix } (f' x))| \leq B$
shows $\text{measurable_bounded_differentiable_image:}$
 $f' S \in \text{lmeasurable}$
and $\text{measure_bounded_differentiable_image:}$
 $\text{measure lebesgue } (f' S) \leq B * \text{measure lebesgue } S \text{ (is ?M)}$

theorem

fixes $f :: \text{real}^n :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n :: _$
assumes $S: S \in \text{sets lebesgue}$
and deriv: $\bigwedge x. x \in S \implies (f \text{ has_derivative } f' x) \text{ (at } x \text{ within } S)$
and int: $(\lambda x. |\det (\text{matrix } (f' x))|) \text{ integrable_on } S$
shows $\text{measurable_differentiable_image: } f' S \in \text{lmeasurable}$
and $\text{measure_differentiable_image:}$
 $\text{measure lebesgue } (f' S) \leq \text{integral } S (\lambda x. |\det (\text{matrix } (f' x))|) \text{ (is ?M)}$

9.32.2 Borel measurable Jacobian determinant**proposition** *borel_measurable_partial_derivatives:*

fixes $f :: \text{real}^m :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n$
assumes $S: S \in \text{sets lebesgue}$
and $f: \bigwedge x. x \in S \implies (f \text{ has_derivative } f' x) \text{ (at } x \text{ within } S)$
shows $(\lambda x. (\text{matrix}(f' x)\$m\$n)) \in \text{borel_measurable } (\text{lebesgue_on } S)$

theorem *borel_measurable_det_Jacobian:*

fixes $f :: \text{real}^n :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n :: _$
assumes $S: S \in \text{sets lebesgue}$ **and** $f: \bigwedge x. x \in S \implies (f \text{ has_derivative } f' x) \text{ (at } x \text{ within } S)$
shows $(\lambda x. \det(\text{matrix}(f' x))) \in \text{borel_measurable } (\text{lebesgue_on } S)$

theorem *borel_measurable_lebesgue_on_preimage_borel:*

fixes $f :: 'a::\text{euclidean_space} \Rightarrow 'b::\text{euclidean_space}$
assumes $S \in \text{sets lebesgue}$
shows $f \in \text{borel_measurable } (\text{lebesgue_on } S) \longleftrightarrow$
 $(\forall T. T \in \text{sets borel} \longrightarrow \{x \in S. f x \in T\} \in \text{sets lebesgue})$

9.32.3 Simplest case of Sard's theorem (we don't need continuity of derivative)

theorem *baby_Sard*:

fixes $f :: \text{real}^m :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n :: \{\text{finite}, \text{wellorder}\}$
assumes $m \leq n$: $\text{CARD}(m) \leq \text{CARD}(n)$
and der : $\bigwedge x. x \in S \implies (f \text{ has_derivative } f' x)$ (at x within S)
and rank : $\bigwedge x. x \in S \implies \text{rank}(\text{matrix}(f' x)) < \text{CARD}(n)$
shows $\text{negligible}(f' S)$

9.32.4 A one-way version of change-of-variables not assuming injectivity.

proposition *absolutely_integrable_on_image*:

fixes $f :: \text{real}^m :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m :: _ \Rightarrow \text{real}^n :: _$
assumes der_g : $\bigwedge x. x \in S \implies (g \text{ has_derivative } g' x)$ (at x within S)
and intS : $(\lambda x. |\det(\text{matrix}(g' x))| *_{\mathbb{R}} f(g x))$ *absolutely_integrable_on* S
shows f *absolutely_integrable_on* $(g' S)$

proposition *integral_on_image_around*:

fixes $f :: \text{real}^n :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}$ **and** $g :: \text{real}^n :: _ \Rightarrow \text{real}^n :: _$
assumes $\bigwedge x. x \in S \implies 0 \leq f(g x)$
and $\bigwedge x. x \in S \implies (g \text{ has_derivative } g' x)$ (at x within S)
and $(\lambda x. |\det(\text{matrix}(g' x))| * f(g x))$ *integrable_on* S
shows $\text{integral}(g' S) f \leq \text{integral } S (\lambda x. |\det(\text{matrix}(g' x))| * f(g x))$

9.32.5 Change-of-variables theorem

theorem *has_absolute_integral_change_of_variables_invertible*:

fixes $f :: \text{real}^m :: \{\text{finite}, \text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m :: _ \Rightarrow \text{real}^n :: _$
assumes der_g : $\bigwedge x. x \in S \implies (g \text{ has_derivative } g' x)$ (at x within S)
and hg : $\bigwedge x. x \in S \implies h(g x) = x$
and conth : *continuous_on* $(g' S) h$
shows $(\lambda x. |\det(\text{matrix}(g' x))| *_{\mathbb{R}} f(g x))$ *absolutely_integrable_on* $S \wedge \text{integral } S (\lambda x. |\det(\text{matrix}(g' x))| *_{\mathbb{R}} f(g x)) = b \iff$
 f *absolutely_integrable_on* $(g' S) \wedge \text{integral}(g' S) f = b$

(is ?lhs = ?rhs)

theorem *has_absolute_integral_change_of_variables_compact*:

fixes $f :: \text{real}^m::\{\text{finite},\text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m::_ \Rightarrow \text{real}^m::_$
assumes *compact S*
and $\text{der}_g: \bigwedge x. x \in S \implies (g \text{ has_derivative } g' x) \text{ (at } x \text{ within } S)$
and $\text{inj}: \text{inj_on } g S$
shows $((\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) \text{ absolutely_integrable_on } S \wedge$
 $\text{integral } S (\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) = b$
 $\longleftrightarrow f \text{ absolutely_integrable_on } (g ' S) \wedge \text{integral } (g ' S) f = b)$

theorem *has_absolute_integral_change_of_variables*:

fixes $f :: \text{real}^m::\{\text{finite},\text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m::_ \Rightarrow \text{real}^m::_$
assumes $S: S \in \text{sets lebesgue}$
and $\text{der}_g: \bigwedge x. x \in S \implies (g \text{ has_derivative } g' x) \text{ (at } x \text{ within } S)$
and $\text{inj}: \text{inj_on } g S$
shows $(\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) \text{ absolutely_integrable_on } S \wedge$
 $\text{integral } S (\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) = b$
 $\longleftrightarrow f \text{ absolutely_integrable_on } (g ' S) \wedge \text{integral } (g ' S) f = b$

corollary *absolutely_integrable_change_of_variables*:

fixes $f :: \text{real}^m::\{\text{finite},\text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m::_ \Rightarrow \text{real}^m::_$
assumes $S \in \text{sets lebesgue}$
and $\bigwedge x. x \in S \implies (g \text{ has_derivative } g' x) \text{ (at } x \text{ within } S)$
and $\text{inj_on } g S$
shows $f \text{ absolutely_integrable_on } (g ' S)$
 $\longleftrightarrow (\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) \text{ absolutely_integrable_on } S$

corollary *integral_change_of_variables*:

fixes $f :: \text{real}^m::\{\text{finite},\text{wellorder}\} \Rightarrow \text{real}^n$ **and** $g :: \text{real}^m::_ \Rightarrow \text{real}^m::_$
assumes $S: S \in \text{sets lebesgue}$
and $\text{der}_g: \bigwedge x. x \in S \implies (g \text{ has_derivative } g' x) \text{ (at } x \text{ within } S)$
and $\text{inj}: \text{inj_on } g S$
and $\text{disj}: (f \text{ absolutely_integrable_on } (g ' S) \vee$
 $(\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x)) \text{ absolutely_integrable_on } S)$
shows $\text{integral } (g ' S) f = \text{integral } S (\lambda x. |\det(\text{matrix}(g' x))| *_R f(g x))$

corollary *absolutely_integrable_change_of_variables_1*:

fixes $f :: \text{real} \Rightarrow \text{real}^n::\{\text{finite},\text{wellorder}\}$ **and** $g :: \text{real} \Rightarrow \text{real}$
assumes $S: S \in \text{sets lebesgue}$
and $\text{der}_g: \bigwedge x. x \in S \implies (g \text{ has_vector_derivative } g' x) \text{ (at } x \text{ within } S)$
and $\text{inj}: \text{inj_on } g S$
shows $(f \text{ absolutely_integrable_on } g ' S \longleftrightarrow$

$(\lambda x. |g' x| *R f(g x))$ *absolutely_integrable_on S*)

9.32.6 Change of variables for integrals: special case of linear function

9.32.7 Change of variable for measure

end

9.33 Lipschitz Continuity

theory *Lipschitz*

imports

Derivative Abstract_Metric_Spaces

begin

definition *lipschitz_on*

where *lipschitz_on C U f* $\longleftrightarrow (0 \leq C \wedge (\forall x \in U. \forall y \in U. \text{dist } (f x) (f y) \leq C * \text{dist } x y))$

notation *lipschitz_on* (*lipschitz'_on* [1000])

proposition *lipschitz_on_uniformly_continuous*:

assumes *L-lipschitz_on X f*

shows *uniformly_continuous_on X f*

proposition *lipschitz_on_continuous_on*:

continuous_on X f if *L-lipschitz_on X f*

proposition *bounded_derivative_imp_lipschitz*:

assumes $\bigwedge x. x \in X \implies (f \text{ has_derivative } f' x)$ (at *x* within *X*)

assumes *convex: convex X*

assumes $\bigwedge x. x \in X \implies \text{onorm } (f' x) \leq C$ $0 \leq C$

shows *C-lipschitz_on X f*

9.33.1 Local Lipschitz continuity

proposition *lipschitz_on_closed_Union*:

assumes $\bigwedge i. i \in I \implies \text{lipschitz_on } M (U i) f$

$\bigwedge i. i \in I \implies \text{closed } (U i)$

finite I

$M \geq 0$

$\{u..(v::\text{real})\} \subseteq (\bigcup i \in I. U i)$

shows *lipschitz_on M {u..v} f*

9.33.2 Local Lipschitz continuity (uniform for a family of functions)

definition *local_lipschitz*:

'a::metric_space set \Rightarrow *'b::metric_space set* \Rightarrow (*'a* \Rightarrow *'b* \Rightarrow *'c::metric_space*) \Rightarrow *bool*

where

local_lipschitz *T X f* $\equiv \forall x \in X. \forall t \in T.$

$\exists u > 0. \exists L. \forall t \in \text{cball } t \ u \cap T. L\text{-lipschitz_on } (\text{cball } x \ u \cap X) (f \ t)$

proposition *c1_implies_local_lipschitz*:

fixes *T::real set* **and** *X::'a::{banach,heine_borel} set*

and *f::real* \Rightarrow *'a* \Rightarrow *'a*

assumes *f'*: $\bigwedge t \ x. t \in T \Longrightarrow x \in X \Longrightarrow (f \ t \text{ has_derivative } \text{blinfun_apply } (f' (t, x))) (at \ x)$

assumes *cont_f'*: *continuous_on* (*T* \times *X*) *f'*

assumes *open T*

assumes *open X*

shows *local_lipschitz* *T X f*

end

theory

Multivariate_Analysis

imports

Ordered_Euclidean_Space

Determinants

Cross3

Lipschitz

Starlike

beginend

9.34 Volume of a Simplex

theory *Simplex_Content*

imports *Change_Of_Vars*

begin

theorem *content_std_simplex*:

measure *lborel* (*convex_hull* (*insert 0 Basis* :: *'a* :: *euclidean_space set*)) = $1 / \text{fact } \text{DIM}('a)$

proposition *measure_lebesgue_linear_transformation*:

fixes *A* :: (*real* \wedge *'n* :: {*finite*, *wellorder*}) *set*

fixes *f* :: $_ \Rightarrow \text{real} \wedge 'n :: \{\text{finite}, \text{wellorder}\}$

assumes *bounded A* *A* \in *sets* *lebesgue* *linear* *f*

shows *measure* *lebesgue* (*f* ' *A*) = $|\text{det } (\text{matrix } f)| * \text{measure } \text{lebesgue } A$

theorem *content_simplex*:

fixes $X :: (\text{real } ^n :: \{\text{finite, wellorder}\}) \text{ set}$ **and** $f :: 'n :: _ \Rightarrow \text{real } ^n :: _)$
assumes $\text{finite } X \text{ card } X = \text{Suc } \text{CARD}(^n)$ **and** $x0: x0 \in X$ **and** $\text{bij: bij_betw } f$
 $\text{UNIV } (X - \{x0\})$
defines $M \equiv (\chi i. \chi j. f j \$ i - x0 \$ i)$
shows $\text{content } (\text{convex hull } X) = |\det M| / \text{fact } (\text{CARD}(^n))$

theorem *content_triangle*:

fixes $A B C :: \text{real } ^2$
shows $\text{content } (\text{convex hull } \{A, B, C\}) =$
 $|(C \$ 1 - A \$ 1) * (B \$ 2 - A \$ 2) - (B \$ 1 - A \$ 1) * (C \$ 2 - A$
 $\$ 2)| / 2$

theorem *heron*:

fixes $A B C :: \text{real } ^2$
defines $a \equiv \text{dist } B C$ **and** $b \equiv \text{dist } A C$ **and** $c \equiv \text{dist } A B$
defines $s \equiv (a + b + c) / 2$
shows $\text{content } (\text{convex hull } \{A, B, C\}) = \text{sqrt } (s * (s - a) * (s - b) * (s -$
 $c))$

end

9.35 Convergence of Formal Power Series

theory *FPS_Convergence*

imports

Generalised_Binomial_Theorem

HOL-Computational_Algebra.Formal_Power_Series

begin

9.35.1 Basic properties of convergent power series

definition $\text{fps_conv_radius} :: 'a :: \{\text{banach, real_normed_div_algebra}\} \text{ fps} \Rightarrow$
 ereal **where**

$\text{fps_conv_radius } f = \text{conv_radius } (\text{fps_nth } f)$

definition $\text{eval_fps} :: 'a :: \{\text{banach, real_normed_div_algebra}\} \text{ fps} \Rightarrow 'a \Rightarrow 'a$
where

$\text{eval_fps } f z = (\sum n. \text{fps_nth } f n * z ^ n)$

theorem *sums_eval_fps*:

fixes $f :: 'a :: \{\text{banach, real_normed_div_algebra}\} \text{ fps}$

assumes $\text{norm } z < \text{fps_conv_radius } f$

shows $(\lambda n. \text{fps_nth } f n * z ^ n) \text{ sums } \text{eval_fps } f z$

9.35.2 Evaluating power series

theorem *eval_fps_deriv*:

assumes $\text{norm } z < \text{fps_conv_radius } f$

shows $eval_fps (fps_deriv f) z = deriv (eval_fps f) z$

theorem *fps_nth_conv_deriv*:

fixes $f :: complex\ fps$

assumes $fps_conv_radius f > 0$

shows $fps_nth f n = (deriv \overset{\sim}{\sim} n) (eval_fps f) 0 / fact n$

theorem *eval_fps_eqD*:

fixes $f g :: complex\ fps$

assumes $fps_conv_radius f > 0\ fps_conv_radius g > 0$

assumes *eventually* $(\lambda z. eval_fps f z = eval_fps g z) (nhds 0)$

shows $f = g$

9.35.3 Power series expansions of analytic functions

definition

has_fps_expansion :: $('a :: \{banach, real_normed_div_algebra\} \Rightarrow 'a) \Rightarrow 'a\ fps$
 $\Rightarrow bool$

(**infixl** *has'_fps'_expansion* 60)

where $(f\ has_fps_expansion\ F) \longleftrightarrow$

$fps_conv_radius F > 0 \wedge eventually (\lambda z. eval_fps F z = f z) (nhds 0)$

end

theory *Smooth_Paths*

imports *Retracts*

begin

9.35.4 Piecewise differentiability of paths

9.35.5 Valid paths, and their start and finish

definition *valid_path* :: $(real \Rightarrow 'a :: real_normed_vector) \Rightarrow bool$

where $valid_path f \equiv f\ piecewise_C1_differentiable_on\ \{0..1::real\}$

end

9.36 Metrics on product spaces

theory *Function_Metric*

imports

Function_Topology

Elementary_Metric_Spaces

begininstantiation *fun* :: $(countable, metric_space)\ metric_space$

begin

definition *dist_fun_def*:

$$\text{dist } x \ y = (\sum n. (1/2)^{\hat{n}} * \min (\text{dist } (x \ (\text{from_nat } n)) \ (y \ (\text{from_nat } n))) \ 1)$$

definition *uniformity_fun_def*:

(*uniformity::('a \Rightarrow 'b) \times ('a \Rightarrow 'b)) filter*) = (*INF* $e \in \{0 < ..\}$. *principal* $\{(x, y).$
dist $(x::('a \Rightarrow 'b)) \ y < e\}$)

end

theory *Analysis*

imports

Convex

Determinants

FSigma

Sum_Topology

Abstract_Topological_Spaces

Abstract_Metric_Spaces

Urysohn

Connected

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