

# The Transfinite Recursion Theorem in Naproche

Marcel Schütz

2025

This is a formalization of the *Transfinite Recursion Theorem* (cf. [1]). It states that for any map  $G: A^{<\infty} \rightarrow A$ , where  $A^{<\infty}$  denotes the class of all maps  $\alpha \rightarrow A$  for some ordinal  $\alpha$ , there exists a unique map  $F: \mathbb{O} \rightarrow A$  that is *recursive regarding*  $G$ , i.e.

$$F(\alpha) = G(F \upharpoonright \alpha)$$

for all ordinals  $\alpha$ .

**Lemma 1 (Coincidence Lemma).** Let  $A$  be a class and  $G$  be a map from  $A^{<\infty}$  to  $A$ . Let  $H, H'$  be maps to  $A$  that are recursive regarding  $G$ . Then

$$H(\alpha) = H'(\alpha)$$

for all  $\alpha \in \text{dom}(H) \cap \text{dom}(H')$ .

*Proof.* Define  $\Phi := \{\alpha \in \mathbb{O} \mid \text{if } \alpha \in \text{dom}(H) \cap \text{dom}(H') \text{ then } H(\alpha) = H'(\alpha)\}$ .

For all ordinals  $\alpha$  if every ordinal less than  $\alpha$  lies in  $\Phi$  then  $\alpha \in \Phi$ .

*Proof.* Let  $\alpha \in \mathbb{O}$ . Assume that every  $y \in \alpha$  lies in  $\Phi$ .

Let us show that if  $\alpha \in \text{dom}(H) \cap \text{dom}(H')$  then  $H(\alpha) = H'(\alpha)$ . Suppose  $\alpha \in \text{dom}(H) \cap \text{dom}(H')$ . Then  $\alpha \subset \text{dom}(H), \text{dom}(H')$ . Indeed  $\text{dom}(H)$  and  $\text{dom}(H')$  are transitive classes. Hence for all  $y \in \alpha$  we have  $y \in \text{dom}(H) \cap \text{dom}(H')$ . Thus  $H(y) = H'(y)$  for all  $y \in \alpha$ . Therefore  $H \upharpoonright \alpha = H' \upharpoonright \alpha$ .  $H$  and  $H'$  are recursive regarding  $G$ . [prover vampire] We have  $H \upharpoonright \alpha, H' \upharpoonright \alpha \in A^{<\infty}$ . Hence  $H(\alpha) = G(H \upharpoonright \alpha) = G(H' \upharpoonright \alpha) = H'(\alpha)$ . End.

Thus  $\alpha \in \Phi$ .  $\square$

[prover vampire] Then  $\Phi$  contains every ordinal (by transfinite induction). Therefore we have  $H(\alpha) = H'(\alpha)$  for all  $\alpha \in \text{dom}(H) \cap \text{dom}(H')$ .  $\blacksquare$

**Theorem 2 (Transfinite Recursion: Existence).** Let  $A$  be a class and  $G$  be a map from  $A^{<\infty}$  to  $A$ . Then there exists a map  $F$  from  $\mathbb{O}$  to  $A$  that is recursive regarding  $G$ .

*Proof.* Every ordinal is contained in the domain of some map  $H$  to  $A$  such that  $H$  is recursive regarding  $G$ .

*Proof.* Define

$$\Phi := \left\{ \alpha \in \mathbb{O} \mid \begin{array}{l} \alpha \text{ is contained in the domain of some map to } A \text{ that is} \\ \text{recursive regarding } G \end{array} \right\}.$$

Let us show that for every ordinal  $\alpha$  if every ordinal less than  $\alpha$  lies in  $\Phi$  then  $\alpha \in \Phi$ . Let  $\alpha$  be an ordinal. Assume that every ordinal less than  $\alpha$  lies in  $\Phi$ . Then for all  $y \in \alpha$  there exists a map  $h$  to  $A$  such that  $h$  is recursive regarding  $G$  and  $y \in \text{dom}(h)$ . Define  $H'(y) :=$  “choose a map  $h$  to  $A$  such that  $h$  is recursive regarding  $G$  and  $y \in \text{dom}(h)$  in  $h(y)$ ” for  $y \in \alpha$ . Then  $H'$  is a map from  $\alpha$  to  $A$ . We have  $H' = H' \upharpoonright \alpha$ .

We can show that  $H' \upharpoonright \beta \in \text{dom}(G)$  for every  $\beta \in \text{succ}(\alpha)$ . Let  $\beta \in \text{succ}(\alpha)$ .  $H' \upharpoonright \beta$  is a function. Indeed  $H' \upharpoonright \beta$  is a map and  $\text{dom}(H' \upharpoonright \beta)$  is a set. Hence  $H' \upharpoonright \beta$  is a function from  $\beta$  to  $A$ . Thus  $H' \upharpoonright \beta \in A^{<\infty}$ . Therefore  $H' \upharpoonright \beta \in \text{dom}(G)$ . End.

Define

$$H(\beta) := \begin{cases} H'(\beta) & : \beta < \alpha \\ G(H' \upharpoonright \beta) & : \beta = \alpha \end{cases}$$

for  $\beta \in \text{succ}(\alpha)$ .

Let us show that  $H \upharpoonright \beta \in A^{<\infty}$  for all  $\beta \in \text{dom}(H)$ . Let  $\beta \in \text{dom}(H)$ .  $\text{dom}(H \upharpoonright \beta) = \beta$  and  $(H \upharpoonright \beta)(b) = H(b)$  for all  $b \in \beta$ .  $H(b) \in A$  for all  $b \in \beta$ . Hence  $H \upharpoonright \beta$  is a map from  $\beta$  to  $A$ . End.

(a)  $\text{dom}(H)$  is a transitive subclass of  $\mathbb{O}$ .

(b) For all  $\beta \in \text{dom}(H)$  we have  $H(\beta) = G(H \upharpoonright \beta)$ .

*Proof.* Let  $\beta \in \text{dom}(H)$ . Then  $\beta < \alpha$  or  $\beta = \alpha$ .

*Case*  $\beta < \alpha$ . Choose a map  $h$  to  $A$  such that  $h$  is recursive regarding  $G$  and  $\beta \in \text{dom}(h)$  and  $H'(\beta) = h(\beta)$ .

Let us show that for all  $y \in \beta$  we have  $h(y) = H(y)$ . Let  $y \in \beta$ . Then  $H(y) = H'(y)$ . Choose a map  $h'$  to  $A$  such that  $h'$  is recursive regarding  $G$  and  $y \in \text{dom}(h')$  and  $H'(y) = h'(y)$ . [prover vampire] Then  $h'(y) = h(y)$  (by coincidence lemma). Indeed  $y \in \text{dom}(h) \cap \text{dom}(h')$ . End.

Hence  $h \restriction \beta = H \restriction \beta$ . Thus  $H(\beta) = H'(\beta) = h(\beta) = G(h \restriction \beta) = G(H \restriction \beta)$ .

□

*Case*  $\beta = \alpha$ . We have  $H \restriction \alpha = H' \restriction \alpha$ . □

□

Hence  $H$  is a map to  $A$  such that  $H$  is recursive regarding  $G$  and  $\alpha \in \text{dom}(H)$ . Thus  $\alpha \in \Phi$ . End.

[prover vampire] Therefore  $\Phi$  contains every ordinal (by transfinite induction). Consequently every ordinal is contained in the domain of some map  $H$  to  $A$  such that  $H$  is recursive regarding  $G$ . □

Define  $F(\alpha) :=$  “choose a map  $H$  to  $A$  such that  $H$  is recursive regarding  $G$  and  $\alpha \in \text{dom}(H)$  in  $H(\alpha)$ ” for  $\alpha \in \mathbb{O}$ . Then  $F$  is a map from  $\mathbb{O}$  to  $A$ .

$F$  is recursive regarding  $G$ .

*Proof.* (a)  $\text{dom}(F)$  is a transitive subclass of  $\mathbb{O}$ .

(b) For all  $\alpha \in \mathbb{O}$  we have  $F(\alpha) = G(F \restriction \alpha)$ .

*Proof.* Let  $\alpha \in \mathbb{O}$ . Choose a map  $H$  to  $A$  such that  $H$  is recursive regarding  $G$  and  $\alpha \in \text{dom}(H)$  and  $F(\alpha) = H(\alpha)$ .

Let us show that  $F(\beta) = H(\beta)$  for all  $\beta \in \alpha$ . Let  $\beta \in \alpha$ . Choose a map  $H'$  to  $A$  such that  $H'$  is recursive regarding  $G$  and  $\beta \in \text{dom}(H')$  and  $F(\beta) = H'(\beta)$ . [prover vampire] Then  $H(\beta) = H'(\beta)$  (by coincidence lemma). Indeed  $\beta \in \text{dom}(H) \cap \text{dom}(H')$ . Therefore  $F(\beta) = H'(\beta)$ . End.

Hence  $H \restriction \alpha = F \restriction \alpha$ . Thus  $F(\alpha) = H(\alpha) = G(H \restriction \alpha) = G(F \restriction \alpha)$ . □ □ ■

**Theorem 3 (Transfinite Recursion: Uniqueness).** Let  $A$  be a class and  $G$  be a map from  $A^{<\infty}$  to  $A$ . Let  $F, F'$  be maps from  $\mathbb{O}$  to  $A$  that are recursive regarding  $G$ . Then  $F = F'$ .

*Proof.* [prover vampire]  $F$  and  $F'$  are recursive regarding  $G$ . Then  $F(\alpha) = F'(\alpha)$  for all  $\alpha \in \text{dom}(F) \cap \text{dom}(F')$  (by coincidence lemma). We have  $\text{dom}(F) = \mathbb{O} = \text{dom}(F')$ . Hence  $F(\alpha) = F'(\alpha)$  for all  $\alpha \in \mathbb{O}$ . Thus  $F = F'$ . ■

As a corollary of the transfinite recursion theorem we get that we can define maps recursively on the ordinals by case distinction: For given maps  $G: \mathbb{O} \times A \rightarrow A$  and  $H: \mathbb{O} \times A^{<\infty} \rightarrow A$  and an element  $a \in A$  we can define a map  $F: \mathbb{O} \rightarrow A$  by

- $F(0) := a$ ,

- $F(\text{succ}(\alpha)) := G(\alpha, F(\alpha))$ , and
- $F(\lambda) := H(\lambda, F \upharpoonright \lambda)$  for any limit ordinal  $\lambda$ .

**Lemma 4.** Let  $A$  be a class and  $\alpha$  be an ordinal and  $F: \mathbb{O} \rightarrow A$ . Then  $(\alpha, F(\alpha)) \in \mathbb{O} \times A$ .

**Lemma 5.** Let  $A$  be a class and  $\lambda$  be a limit ordinal and  $F: \mathbb{O} \rightarrow A$ . Then  $(\lambda, F \upharpoonright \lambda) \in \mathbb{O} \times A^{<\infty}$ .

**Corollary 6.** Let  $A$  be a class. Let  $a \in A$  and  $G: \mathbb{O} \times A \rightarrow A$  and  $H: \mathbb{O} \times A^{<\infty} \rightarrow A$ . Then there exists a map  $F$  from  $\mathbb{O}$  to  $A$  such that

$$F(0) = a$$

and for all ordinals  $\alpha$  we have

$$F(\text{succ}(\alpha)) = G(\alpha, F(\alpha))$$

and for all limit ordinals  $\lambda$  we have

$$F(\lambda) = H(\lambda, F \upharpoonright \lambda).$$

*Proof.*  $(\text{pred}(\text{dom}(f)), f(\text{pred}(\text{dom}(f)))) \in \mathbb{O} \times A$  for all  $f \in A^{<\infty}$  such that  $\text{dom}(f)$  is a successor ordinal.

Define

$$J(f) := \begin{cases} a & : \text{dom}(f) = 0 \\ G(\text{pred}(\text{dom}(f)), f(\text{pred}(\text{dom}(f)))) & : \text{dom}(f) \text{ is a successor ordinal} \\ H(\text{dom}(f), f) & : \text{dom}(f) \text{ is a limit ordinal} \end{cases}$$

for  $f \in A^{<\infty}$ .

Then  $J$  is a map from  $A^{<\infty}$  to  $A$ . Indeed we can show that for any  $f \in A^{<\infty}$  we have  $J(f) \in A$ . Let  $f \in A^{<\infty}$ . Take  $\alpha \in \mathbb{O}$  such that  $f: \alpha \rightarrow A$ . If  $\alpha = 0$  then  $J(f) = a \in A$ . If  $\alpha$  is a successor ordinal then  $J(f) = G(\text{pred}(\alpha), f(\text{pred}(\alpha))) \in A$ . [prover vampire] If  $\alpha$  is a limit ordinal then  $J(f) = H(\alpha, f) \in A$ . End.

Hence we can take a map  $F$  from  $\mathbb{O}$  to  $A$  that is recursive regarding  $J$ . Then  $F \upharpoonright \alpha \in A^{<\infty}$  for any ordinal  $\alpha$ .

(1)  $F(0) = a$ .

*Proof.*  $F(0) = J(F \upharpoonright 0) = a$ .  $\square$

(2)  $F(\text{succ}(\alpha)) = G(\alpha, F(\alpha))$  for all ordinals  $\alpha$ .

*Proof.* Let  $\alpha$  be an ordinal. Then  $F(\text{succ}(\alpha)) = J(F \upharpoonright \text{succ}(\alpha)) = G(\text{pred}(\text{succ}(\alpha)), (F \upharpoonright \text{succ}(\alpha))(\text{pred}(\text{succ}(\alpha)))) = G(\alpha, (F \upharpoonright \text{succ}(\alpha))(\alpha)) = G(\alpha, F(\alpha))$ .  $\square$

(3)  $F(\lambda) = H(\lambda, F \upharpoonright \lambda)$  for all limit ordinals  $\lambda$ .

*Proof.* Let  $\lambda$  be a limit ordinal. Then  $F(\lambda) = J(F \upharpoonright \lambda) = H(\lambda, F \upharpoonright \lambda)$ .  $\square$

Hence the thesis (by 1, 2, 3).  $\blacksquare$

## References

- [1] Peter Koepke. “Set Theory”. Lecture notes. 2018/19. URL: [http://www.math.uni-bonn.de/ag/logik/teaching/2018WS/set\\_theory/current\\_scriptum.pdf](http://www.math.uni-bonn.de/ag/logik/teaching/2018WS/set_theory/current_scriptum.pdf).

## License

© Marcel Schütz (2024–2025). This work is licensed under a [Creative Commons “Attribution-NonCommercial-ShareAlike 4.0 International”](#) (CC BY-NC-SA 4.0) license.