

König's Theorem

Naproche formalization:

Steffen Frerix and Peter Koepke
University of Bonn

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König's Theorem is an important set-theoretical result about the arithmetic of cardinals. It was proved by Julius König in 1905 [Koenig1905]. The proof is reminiscent of Cantor's diagonal argument for proving that $\kappa < 2^\kappa$.

Sums and Products of cardinals

Let f_i stand for $f(i)$. Let D denote a set.

Definition. A sequence of cardinals on D is a function κ such that $\text{dom}(\kappa) = D$ and κ_i is a cardinal for every element i of D .

Definition. Let κ be a sequence of cardinals on D .

$$\bigsqcup_{i \in D} \kappa_i = \{(n, i) \mid i \text{ is an element of } D \text{ and } n \text{ is an element of } \kappa_i\}.$$

Axiom. Let κ be a sequence of cardinals on D . Then $\bigsqcup_{i \in D} \kappa_i$ is a set.

Definition. Let κ be a sequence of cardinals on D .

$$\sum_{i \in D} \kappa_i = \left| \bigsqcup_{i \in D} \kappa_i \right|.$$

Definition. Let κ be a sequence of cardinals on D .

$$\prod_{i \in D} \kappa_i = \left\{ f \mid f \text{ is a function and } \text{dom}(f) = D \text{ and } f(i) \text{ is an element of } \kappa_i \text{ for every element } i \text{ of } D \right\}.$$

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$$\prod_{i \in D} \kappa_i = \left| \times_{i \in D} \kappa_i \right|.$$

König's Theorem requires some form of the axiom of choice. Currently choice is built into Naproche by the *choose* construct in function definitions. The axiom of choice is also required to show that products of non-empty factors are themselves non-empty:

Lemma (Choice). Let λ be a sequence of cardinals on D . Assume that λ_i has an element for every element i of D . Then $\times_{i \in D} \lambda_i$ has an element. *Proof.* Define $f(i) = \text{“choose an element } v \text{ of } \lambda_i \text{ in } v\text{”}$ for i in D . Then f is an element of $\times_{i \in D} \lambda_i$. \square

König's theorem

Theorem (König). Let κ, λ be sequences of cardinals on D . Assume that for every element i of D $\kappa_i < \lambda_i$. Then

$$\sum_{i \in D} \kappa_i < \prod_{i \in D} \lambda_i.$$

Proof by contradiction. Assume the contrary. Then

$$\prod_{i \in D} \lambda_i \leq \sum_{i \in D} \kappa_i.$$

Take a surjective map G from $\bigsqcup_{i \in D} \kappa_i$ to $\times_{i \in D} \lambda_i$. Indeed $\times_{i \in D} \lambda_i$ and $\sum_{i \in D} \kappa_i$ are nonempty sets. Take $\Lambda = \bigcup \text{range}(G)$. Then Λ is a set. Indeed $\text{range}(G)$ is a set.

$(n, i) \in \text{dom}(G)$ for every $i \in D$ and every $n \in \kappa_i$. $G(n, i) \in \times_{i \in D} \lambda_i$ for every $i \in D$ and every $n \in \kappa_i$. Hence for every $i \in D$ and every $n \in \kappa_i$ $G(n, i)$ is a map such that $i \in \text{dom}(G(n, i))$.

Define $\Delta(i) = \{G(n, i)(i) \in \Lambda \mid n \in \kappa_i\}$ for $i \in D$.

For every element f of $\times_{i \in D} \lambda_i$ and every element i of D we have $f(i) \in \Lambda$.

For every element i of D we have $|\Delta(i)| < \lambda_i$.

Proof. Let i be an element of D . Define $F(n) = G(n, i)(i)$ for n in κ_i . Then F is a map from κ_i to λ_i . We have $\Delta(i) = \{F(n) \mid n \in \kappa_i\}$. Thus $F[\kappa_i] = \Delta(i)$. Therefore $|\Delta(i)| = |F[\kappa_i]| \leq |\kappa_i| = \kappa_i < \lambda_i$. Indeed $|F[\kappa_i]| \leq |\kappa_i|$ (by SET_THEORY_06_8113916590686208). Indeed κ_i and λ_i are sets. End.

Define $f(i) = \text{“choose an element } v \text{ of } \lambda_i \setminus \Delta(i) \text{ in } v\text{”}$ for $i \in D$. Indeed $\lambda_i \setminus \Delta(i)$ is nonempty for each $i \in D$. Then f is an element of $\times_{i \in D} \lambda_i$. Take an element j of D and an element m of κ_j such that $G(m, j) = f$. $G(m, j)(j)$ is an element of $\Delta(j)$ and $f(j)$ is not an element of $\Delta(j)$. Contradiction. \square