

Cantor's Paradoxes

Naproche formalization:

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Cantor's First Paradox, discovered by Georg Cantor in 1897, states that the collection **Card** of all (infinite) cardinal numbers is a proper class [Cantor1991].

Theorem (Cantor's First Paradox). **Card** is a proper class.

Proof. Suppose that **Card** is a set. Then $\bigcup \mathbf{Card}$ is a set.

Let us show that $\bigcup \mathbf{Card}$ contains every ordinal. Let α be an ordinal. Choose an infinite ordinal β such that $\beta \geq \alpha$. Choose a cardinal κ greater than β . Then $\alpha \in \kappa \in \mathbf{Card}$. Hence $\alpha \in \bigcup \mathbf{Card}$. End.

Therefore $\mathbf{Ord} \subseteq \bigcup \mathbf{Card}$. Thus **Ord** is a set. Contradiction. \square

Cantor's Second Paradox denotes the observation that the collection **V** of all set is a proper class. It was shown by Georg Cantor in 1899 via his famous theorem stating that the cardinality of any set is strictly smaller than the cardinality of its powerset [Cantor1991].

Theorem (Cantor's Second Paradox). **V** is a proper class.

Proof. Assume the contrary. Then $\mathcal{P}(\mathbf{V})$ is a set. Hence $\mathcal{P}(\mathbf{V})$ is a subset of **V**. Thus $|\mathcal{P}(\mathbf{V})| \leq |\mathbf{V}|$. Contradiction. Indeed $|x| < |\mathcal{P}(x)|$ for any set x . \square

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