Closure Under Finite Unions

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Definition 0.1. Let X be a system of sets. X is closed under finite unions iff $\bigcup U \in X$ for every nonempty finite subclass U of X.

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Proposition 0.2. Let X be a system of sets. X is closed under finite unions iff $U \cup V \in X$ for every $U, V \in X$.

Proof. Case X is closed under finite unions. Let $U, V \in X$. Then $\{U, V\}$ is a nonempty finite subclass of X. Hence $U \cup V = \bigcup\{U, V\} \in X$. End.

Case $U \cup V \in X$ for every $U, V \in X$. Define $\Phi = \{n \in \mathbb{N} \mid \bigcup U \in X$ for every nonempty subclass U of X that has n elements $\}$.

(1) Φ contains 0.

(2) For every $n \in \Phi$ we have $n + 1 \in \Phi$.
Proof. Let $n \in \Phi$. Then $\bigcup U \in X$ for every nonempty subclass U of X that has n elements.

Let us show that $\bigcup U \in X$ for every nonempty subclass U of X that has $n + 1$ elements.

Case $n = 0$. Obvious.

Case $n \neq 0$. Let U be a nonempty subclass of X such that U has $n + 1$ elements. Take a bijection f between $\{1, \ldots, n + 1\}$ and U. We have $\{1, \ldots, n + 1\} = \{1, \ldots, n\} \cup \{n + 1\}$. Take $V = f[\{1, \ldots, n\}]$. We have $\{1, \ldots, n\} \subseteq \{1, \ldots, n + 1\}$.

Let us show that $V \subseteq U$. Let $x \in V$. Take $k \in \{1, \ldots, n\}$ such that $x = f(k)$. Indeed we can show that there exists a $k \in \{1, \ldots, n\}$ such that $x = f(k)$. Assume the contrary. Then $x \neq f(k)$ for all $k \in \{1, \ldots, n\}$. Hence $x \notin f[\{1, \ldots, n\}] = V$. Contradiction. End. Hence $x \in U$. Indeed $x \in f[\{1, \ldots, n + 1\}]$. End.

V is nonempty. Indeed $f(1) \in f[\{1, \ldots, n\}]$. Indeed $1 \in \{1, \ldots, n\}$. Hence $f \restriction \{1, \ldots, n\}$ is a bijection between $\{1, \ldots, n\}$ and V (by **??**). Thus V has n elements. Consequently $\bigcup V \in X$.

Let us show that $U = V \cup \{f(n + 1)\}$.
(1) $f[A \cup B] = f[A] \cup f[B]$ for all $A, B \subseteq \mathrm{dom}(f)$.

(2) $f[\{a\}] = \{f(a)\}$ for all $a \in \mathrm{dom}(f)$.

Hence $U = f[\mathrm{dom}(f)] = f[\{1, \ldots, n + 1\}] = f[\{1, \ldots, n\} \cup \{n + 1\}] = f[\{1, \ldots, n\}] \cup f[\{n + 1\}] = f[\{1, \ldots, n\}] \cup \{f(n + 1)\} = V \cup \{f(n + 1)\}$.

Indeed $n+1 \in \mathrm{dom}(f)$ and $\{1,\ldots,n\},\{1+n\} \subseteq \mathrm{dom}(f)$. End.

Let us show that $\bigcup(A \cup B) = (\bigcup A) \cup (\bigcup B)$ for any nonempty systems of sets A, B. Let A, B be nonempty systems of sets. $\bigcup(A \cup B) \subseteq (\bigcup A) \cup (\bigcup B)$. $((\bigcup A) \cup (\bigcup B)) \subseteq \bigcup(A \cup B)$. End.

$f(n+1)$ and $f(k)$ are sets for every $k \in \{1,\ldots,n\}$. Indeed $f(k) \in U$ for every $k \in \{1,\ldots,n\}$. Hence V and $\{f(n+1)\}$ are nonempty systems of sets. Thus $\bigcup U = \bigcup(V \cup \{f(n+1)\}) = (\bigcup V) \cup (\bigcup\{f(n+1)\}) = (\bigcup V) \cup f(n+1) \in X$. End. End. Qed.

Therefore Φ contains every natural number (by `ARITHMETIC_01_4764664342773760`). Thus $\bigcup U \in X$ for every nonempty finite subclass U of X. Consequently X is closed under finite unions. End. $\square$