Multiplication

Definition

ARITHMETIC_06_6626346484629504

Signature 0.1. Let n, m be natural numbers. $n \cdot m$ is a natural number.

Let the product of n and m stand for $n \cdot m$.

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Axiom 0.2. Let n be a natural number. Then $n \cdot 0 = 0$.

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Axiom 0.3. Let n, m be natural numbers. Then $n \cdot (m + 1) = (n \cdot m) + n$.

Computation Laws

Distributivity

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Proposition 0.4. Let n, m, k be natural numbers. Then $n \cdot (m + k) = (n \cdot m) + (n \cdot k)$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n \cdot (m + k') = (n \cdot m) + (n \cdot k')\}$.

(1) 0 is an element of Φ. Indeed $n \cdot (m + 0) = n \cdot m = (n \cdot m) + 0 = (n \cdot m) + (n \cdot 0)$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then

$$n \cdot (m + (k' + 1))$$

$$= n \cdot ((m + k') + 1)$$

$$= (n \cdot (m + k')) + n$$

$$= ((n \cdot m) + (n \cdot k')) + n$$

$$= (n \cdot m) + ((n \cdot k') + n)$$

$$= (n \cdot m) + (n \cdot (k' + 1)).$$

Hence $n \cdot (m + (k' + 1)) = (n \cdot m) + (n \cdot (k' + 1))$. Thus $k' + 1 \in \Phi$. Qed.

Thus every natural number is contained in Φ. Therefore $n \cdot (m + k) = (n \cdot m) + (n \cdot k)$. ☐

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Proposition 0.5. Let n, m, k be natural numbers. Then $(n+m) \cdot k = (n \cdot k) + (m \cdot k)$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid (n + m) \cdot k' = (n \cdot k') + (m \cdot k')\}$.

(1) 0 belongs to Φ. Indeed $(n + m) \cdot 0 = 0 = 0 + 0 = (n \cdot 0) + (m \cdot 0)$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then

$$\begin{aligned}
&(n + m) \cdot (k' + 1) \\
&= ((n + m) \cdot k') + (n + m) \\
&= ((n \cdot k') + (m \cdot k')) + (n + m) \\
&= (((n \cdot k') + (m \cdot k')) + n) + m \\
&= ((n \cdot k') + ((m \cdot k') + n)) + m \\
&= ((n \cdot k') + (n + (m \cdot k'))) + m \\
&= (((n \cdot k') + n) + (m \cdot k')) + m \\
&= ((n \cdot k') + n) + ((m \cdot k') + m) \\
&= (n \cdot (k' + 1)) + (m \cdot (k' + 1)).
\end{aligned}$$

Thus $(n + m) \cdot (k' + 1) = (n \cdot (k' + 1)) + (m \cdot (k' + 1))$. Qed.

Thus every natural number is an element of Φ. Therefore $(n + m) \cdot k = (n \cdot k) + (m \cdot k)$. ☐

Multiplication with 1 and 2

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Proposition 0.6. Let n be a natural number. Then $n \cdot 1 = n$.

Proof. $n \cdot 1 = n \cdot (0 + 1) = (n \cdot 0) + n = 0 + n = n$. ☐

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Corollary 0.7. Let n be a natural number. Then $n \cdot 2 = n + n$.

Proof. $n \cdot 2 = n \cdot (1+1) = (n \cdot 1) + n = n + n$. ☐

Associativity

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Proposition 0.8. Let n, m, k be natural numbers. Then $n \cdot (m \cdot k) = (n \cdot m) \cdot k$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n \cdot (m \cdot k') = (n \cdot m) \cdot k'\}$.

(1) 0 is contained in Φ. Indeed $n \cdot (m \cdot 0) = n \cdot 0 = 0 = (n \cdot m) \cdot 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then

$$\begin{aligned} & n \cdot (m \cdot (k'+1)) \\ &= n \cdot ((m \cdot k') + m) \\ &= (n \cdot (m \cdot k')) + (n \cdot m) \\ &= ((n \cdot m) \cdot k') + (n \cdot m) \\ &= ((n \cdot m) \cdot k') + ((n \cdot m) \cdot 1) \\ &= (n \cdot m) \cdot (k'+1). \end{aligned}$$

Qed.

Hence every natural number is contained in Φ (by `ARITHMETIC_01_4764664342773760`). Thus $n \cdot (m \cdot k) = (n \cdot m) \cdot k$. ☐

Commutativity

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Proposition 0.9. Let n, m be natural numbers. Then

$$n \cdot m = m \cdot n.$$

Proof. Define $\Phi = \{m' \in \mathbb{N} \mid n \cdot m' = m' \cdot n\}$.

(1) 0 is contained in Φ.
Proof. Define $\Psi = \{n' \in \mathbb{N} \mid n' \cdot 0 = 0 \cdot n'\}$.

(1a) 0 is contained in Ψ.

(1b) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$. Then

$$(n'+1) \cdot 0 = 0 = n' \cdot 0 = 0 \cdot n' = (0 \cdot n') + 0 = 0 \cdot (n'+1).$$

Qed.

Hence every natural number is contained in Ψ (by `ARITHMETIC_01_4764664342773760`). Thus $n \cdot 0 = 0 \cdot n$. Qed.

(2) 1 belongs to Φ.
Proof. Define $\Theta = \{n' \in \mathbb{N} \mid n' \cdot 1 = 1 \cdot n'\}$.

(2a) 0 is contained in Θ.

(2b) For all $n' \in \Theta$ we have $n'+1 \in \Theta$.
Proof. Let $n' \in \Theta$. Then

$$\begin{aligned} &(n'+1) \cdot 1 \\ &= (n' \cdot 1) + 1 \\ &= (1 \cdot n') + 1 \\ &= 1 \cdot (n'+1). \end{aligned}$$

Qed.

Thus every natural number is contained in Θ (by `ARITHMETIC_01_4764664342773760`). Therefore $n \cdot 1 = 1 \cdot n$. Qed.

(3) For all $m' \in \Phi$ we have $m'+1 \in \Phi$.
Proof. Let $m' \in \Phi$. Then

$$\begin{aligned} &n \cdot (m'+1) \\ &= (n \cdot m') + (n \cdot 1) \\ &= (m' \cdot n) + (1 \cdot n) \\ &= (1 \cdot n) + (m' \cdot n) \\ &= (1 + m') \cdot n \\ &= (m'+1) \cdot n. \end{aligned}$$

Indeed $((1 \cdot n) + (m' \cdot n)) = (1 + m') \cdot n$. Qed.

Hence every natural number is contained in Φ (by `ARITHMETIC_01_4764664342773760`). Thus $n \cdot m = m \cdot n$. □

Non-Existence of Zero-Divisors

ARITHMETIC_06_3843962875936768

Proposition 0.10. Let n, m be natural numbers such that $n \cdot m = 0$. Then $n = 0$ or $m = 0$.

Proof. Suppose $n, m \neq 0$. Take natural numbers n', m' such that $n = (n'+1)$ and $m = (m'+1)$. Then

$$0$$

$$= n \cdot m$$
$$= (n'+1) \cdot (m'+1)$$
$$= ((n'+1) \cdot m') + (n'+1)$$
$$= (((n'+1) \cdot m') + n') + 1.$$

Indeed $(n'+1) \cdot (m'+1) = ((n'+1) \cdot m') + (n'+1)$. Hence $0 = k + 1$ for some natural number k. Contradiction. □

Cancellation

ARITHMETIC_06_31055184658432

Proposition 0.11. Let n, m, k be natural numbers. Assume $k \neq 0$. If $n \cdot k = m \cdot k$ then $n = m$.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid$ for all $m' \in \mathbb{N}$ if $n' \cdot k = m' \cdot k$ and $k \neq 0$ then $n' = m'\}$.

(1) 0 is contained in Φ.
Proof. Let $m' \in \mathbb{N}$. Assume $0 \cdot k = m' \cdot k$ and $k \neq 0$. Then $m' \cdot k = 0$. Hence $m' = 0$ or $k = 0$. Thus $m' = 0$. Qed.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.
Proof. Let $n' \in \Phi$.

Let us show that for all $m' \in \mathbb{N}$ if $(n'+1) \cdot k = m' \cdot k$ and $k \neq 0$ then $n' + 1 = m'$. Let $m' \in \mathbb{N}$. Assume $(n'+1) \cdot k = m' \cdot k$ and $k \neq 0$.

Case $m' = 0$. Then $(n'+1) \cdot k = 0$. Hence $n' + 1 = 0$. Contradiction. End.

Case $m' \neq 0$. Take a natural number l such that $m' = l + 1$. Then $(n'+1) \cdot k = (l+1) \cdot k$. Hence $(n' \cdot k) + k = (n' \cdot k) + (1 \cdot k) = (n' \cdot k) + k = (l+1) \cdot k = (l \cdot k) + (1 \cdot k) = (l \cdot k) + k$. Thus $n' \cdot k = l \cdot k$ (by `ARITHMETIC_03_3137702874578944`). Indeed $n' \cdot k$ and $l \cdot k$ are natural numbers. Then we have $n' = l$. Indeed if $n' \cdot k = l \cdot k$ and $k \neq 0$ then $n' = l$. Therefore $n' + 1 = l + 1 = m'$. End. End.

Hence $n' + 1 \in \Phi$. Qed.

Thus every natural number is contained in Φ (by `ARITHMETIC_01_4764664342773760`). Therefore if $n \cdot k = m \cdot k$ then $n = m$. □

ARITHMETIC_06_8575191374364672

Corollary 0.12. Let n, m, k be natural numbers. Assume $k \neq 0$. If $k \cdot n = k \cdot m$ then $n = m$.

Proof. Assume $k \cdot n = k \cdot m$. We have $k \cdot n = n \cdot k$ and $k \cdot m = m \cdot k$. Hence $n \cdot k = m \cdot k$. Thus $n = m$ (by ARITHMETIC_06_31055184658432). □