

Dedekind's Recursion Theorem

Naproche formalization:

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This is a formalization of Dedekind's recursion theorem. It states that for any function $f : A \rightarrow A$ and any $a \in A$ there exists a unique function $\varphi : \mathbb{N} \rightarrow A$ that is *recursively defined* by f and a , i.e. $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$. (cf. [Ebbinghaus1991]).

Definition. Let a be an object and f be a map. Let φ be a map from \mathbb{N} to $\text{dom}(f)$. φ is recursively defined by a and f iff $\varphi(0) = a$ and $\varphi(n+1) = f(\varphi(n))$ for every $n \in \mathbb{N}$.

Theorem (Dedekind's Recursion Theorem: Existence). Let A be a set and $a \in A$ and $f : A \rightarrow A$. Then there exists a $\varphi : \mathbb{N} \rightarrow A$ that is recursively defined by a and f .

Proof. (a) Define

$$\Phi = \left\{ H \in \mathcal{P}(\mathbb{N} \times A) \mid \begin{array}{l} (0, a) \in H \text{ and for all } n \in \mathbb{N} \text{ and all } x \in A \text{ if} \\ (n, x) \in H \text{ then } (n+1, f(x)) \in H \end{array} \right\}.$$

Let us show that $\bigcap \Phi \in \Phi$.

Proof. (0) $\mathbb{N} \times A \in \Phi$.

Proof.

(i) $\mathbb{N} \times A \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in \mathbb{N} \times A$.

(iii) for all $n \in \mathbb{N}$ and all $x \in A$ if $(n, x) \in \mathbb{N} \times A$ then $(n+1, f(x)) \in \mathbb{N} \times A$.

Proof. Let $n \in \mathbb{N}$ and $x \in A$. Assume $(n, x) \in \mathbb{N} \times A$. We have $n+1 \in \mathbb{N}$ and $f(x) \in A$. Hence $(n+1, f(x)) \in \mathbb{N} \times A$. Qed. Qed.

(1) $\bigcap \Phi \in \mathcal{P}(\mathbb{N} \times A)$.

Proof. Any element of $\bigcap \Phi$ is contained in every element of Φ . Hence any element of $\bigcap \Phi$ is contained in $\mathbb{N} \times A$. Thus $\bigcap \Phi \subseteq \mathbb{N} \times A$. $\bigcap \Phi$ is a set. Hence $\bigcap \Phi$ is a subset of $\mathbb{N} \times A$. Qed.

(2) $(0, a) \in \bigcap \Phi$.

Indeed $(0, a) \in \mathbb{N} \times A \in \Phi$.

(3) For all $n \in \mathbb{N}$ and all $x \in A$ if $(n, x) \in \bigcap \Phi$ then $(n + 1, f(x)) \in \bigcap \Phi$.

Proof. Let $n \in \mathbb{N}$ and $x \in A$. Assume $(n, x) \in \bigcap \Phi$. Then (n, x) is contained in every element of Φ . Hence $(n + 1, f(x))$ is contained in every element of Φ . Thus $(n + 1, f(x)) \in \bigcap \Phi$. Qed.

Therefore $\bigcap \Phi \in \Phi$ (by a). Qed.

Let us show that for any $n \in \mathbb{N}$ there exists an $x \in A$ such that $(n, x) \in \bigcap \Phi$.

Proof. Define $\Psi = \{n \in \mathbb{N} \mid \text{there exists an } x \in A \text{ such that } (n, x) \in \bigcap \Phi\}$.

(1) 0 is contained in Ψ . Indeed $(0, a) \in \bigcap \Phi$.

(2) For all $n \in \Psi$ we have $n + 1 \in \Psi$.

Proof. Let $n \in \Psi$. Take an $x \in A$ such that $(n, x) \in \bigcap \Phi$. Then $(n + 1, f(x)) \in \bigcap \Phi$. Hence $n + 1 \in \Psi$. Indeed $f(x) \in A$. Qed.

Therefore $n \in \Psi$ for every $n \in \mathbb{N}$ (by ARITHMETIC_01_4764664342773760). Qed.

Let us show that for all $n \in \mathbb{N}$ and all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$.

Proof. (b) Define $\Theta = \{n \in \mathbb{N} \mid \text{for all } x, y \in A \text{ if } (n, x), (n, y) \in \bigcap \Phi \text{ then } x = y\}$.

(1) Θ contains 0.

Proof. Let us show that for all $x, y \in A$ if $(0, x), (0, y) \in \bigcap \Phi$ then $x = y$. Let $x, y \in A$. Assume $(0, x), (0, y) \in \bigcap \Phi$.

Let us show that $x, y = a$. Assume $x \neq a$ or $y \neq a$.

Case $x \neq a$. $(0, x), (0, a)$ are contained in every element of Φ . Then $(0, x), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, x)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(n + 1, f(b)) \in H$.

Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(n + 1, f(b)) \in \bigcap \Phi$. We have $(n + 1, f(b)) \neq (0, x)$ (by FOUNDATIONS_04_8464577431863296). Indeed $n + 1 \neq 0$. Hence $(n + 1, f(b)) \in H$. Qed.

[prover vampire] Thus $H \in \Phi$ (by a). [prover eprover] End.

Then $(0, x)$ is not contained in every member of Φ . Contradiction. End.

Case $y \neq a$. $(0, y), (0, a)$ are contained in every element of Φ . Then $(0, y), (0, a) \in \bigcap \Phi$. Take $H = (\bigcap \Phi) \setminus \{(0, y)\}$.

Let us show that $H \in \Phi$. (1) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(2) $(0, a) \in H$. Indeed $(0, a) \neq (0, y)$.

(3) For all $n \in \mathbb{N}$ and all $b \in A$ if $(n, b) \in H$ then $(n + 1, f(b)) \in H$.

Proof. Let $n \in \mathbb{N}$ and $b \in A$. Assume $(n, b) \in H$. Then $(n+1, f(b)) \in \bigcap \Phi$. We have $(n+1, f(b)) \neq (0, y)$ (by FOUNDATIONS_04_8464577431863296). Indeed $n+1 \neq 0$. Hence $(n+1, f(b)) \in H$. Qed.

[prover vampire] Thus $H \in \Phi$ (by a). [prover eprover] End.

Then $(0, y)$ is not contained in every member of Φ . Contradiction. End. End. End. Qed.

(2) For all $n \in \Theta$ we have $n+1 \in \Theta$.

Proof. Let $n \in \Theta$. Then for all $x, y \in A$ if $(n, x), (n, y) \in \bigcap \Phi$ then $x = y$. Consider a $b \in A$ such that $(n, b) \in \bigcap \Phi$. Then $(n+1, f(b)) \in \bigcap \Phi$.

Let us show that for all $x, y \in A$ if $(n+1, x), (n+1, y) \in \bigcap \Phi$ then $x = f(b) = y$. Let $x, y \in A$. Assume $(n+1, x), (n+1, y) \in \bigcap \Phi$. Suppose $x \neq f(b)$ or $y \neq f(b)$.

Case $x \neq f(b)$. Take $H = (\bigcap \Phi) \setminus \{(n+1, x)\}$.

(i) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in H$. Indeed $(0, a) \in \bigcap \Phi$ and $(0, a) \notin \{(n+1, x)\}$.

(iii) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(m+1, f(z)) \in H$.

Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(m+1, f(z)) \in \bigcap \Phi$. $(m+1, f(z)) \neq (n+1, x)$ (by FOUNDATIONS_04_8464577431863296). Indeed if $(m+1, f(z)) = (n+1, x)$ then $m = n$ and $f(z) = x$ (by FOUNDATIONS_04_8464577431863296, ARITHMETIC_01_3604163883696128). Therefore $(m+1, f(z)) \in H$. Qed.

[prover vampire] Thus $H \in \Phi$ (by a, i, ii, iii). [prover eprover] Therefore every element of $\bigcap \Phi$ is contained in H . Consequently $(n+1, x) \in H$. Contradiction. End.

Case $y \neq f(b)$. Take $H = (\bigcap \Phi) \setminus \{(n+1, y)\}$.

(i) $H \in \mathcal{P}(\mathbb{N} \times A)$.

(ii) $(0, a) \in H$. Indeed $(0, a) \in \bigcap \Phi$ and $(0, a) \notin \{(n+1, y)\}$.

(iii) For all $m \in \mathbb{N}$ and all $z \in A$ if $(m, z) \in H$ then $(m+1, f(z)) \in H$.

Proof. Let $m \in \mathbb{N}$ and $z \in A$. Assume $(m, z) \in H$. Then $(m, z) \in \bigcap \Phi$. Hence $(m+1, f(z)) \in \bigcap \Phi$. $(m+1, f(z)) \neq (n+1, y)$ (by FOUNDATIONS_04_8464577431863296). Indeed if $(m+1, f(z)) = (n+1, y)$ then $m = n$ and $f(z) = y$ (by FOUNDATIONS_04_8464577431863296, ARITHMETIC_01_3604163883696128). Therefore $(m+1, f(z)) \in H$. Qed.

[prover vampire] Thus $H \in \Phi$ (by a, i, ii, iii). [prover eprover] Therefore every element of $\bigcap \Phi$ is contained in H . Consequently $(n+1, y) \in H$. Contradiction. End.

Hence it is wrong that $x \neq f(b)$ or $y \neq f(b)$. Consequently $x = f(b) = y$. End.

Therefore $n + 1 \in \Theta$ (by a). Qed.

Consequently $n \in \Theta$ for every $n \in \mathbb{N}$ (by ARITHMETIC_01_476466434277 3760). Qed.

Define $\varphi(n) =$ “choose $x \in A$ such that $(n, x) \in \bigcap \Phi$ in x ” for $n \in \mathbb{N}$.

(1) Then φ is a map from \mathbb{N} to A and we have $\varphi(0) = a$.

(2) For all $n \in \mathbb{N}$ we have $\varphi(n + 1) = f(\varphi(n))$.

Proof. Let $n \in \mathbb{N}$. Take $x \in A$ such that $\varphi(n) = x$. Then $(n, x) \in \bigcap \Phi$. Hence $(n + 1, f(\varphi(n))) = (n + 1, f(x)) \in \bigcap \Phi$. Thus $\varphi(n + 1) = f(\varphi(n))$ (by a). Qed. \square

Theorem (Dedekind’s Recursion Theorem: Uniqueness). Let A be a set and $a \in A$ and $f : A \rightarrow A$. Let $\varphi, \varphi' : \mathbb{N} \rightarrow A$. Assume that φ and φ' are recursively defined by a and f . Then $\varphi = \varphi'$.

Proof. Define $\Phi = \{n \in \mathbb{N} \mid \varphi(n) = \varphi'(n)\}$.

(1) Φ contains 0. Indeed $\varphi(0) = a = \varphi'(0)$.

(2) For all $n \in \Phi$ we have $n + 1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $\varphi(n) = \varphi'(n)$. Hence $\varphi(n + 1) = f(\varphi(n)) = f(\varphi'(n)) = \varphi'(n + 1)$. Qed.

Thus Φ contains every natural number. Consequently $\varphi(n) = \varphi'(n)$ for each $n \in \mathbb{N}$. \square

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