Natural Numbers

The Language of Natural Number Arithmetic

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Signature 0.1. A natural number is an object.

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Definition 0.2. $\mathbb{N}$ is the class of natural numbers.

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Signature 0.3. Let n, m be natural numbers. $n + m$ is a natural number.

Let the sum of n and m stand for $n + m$.

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Signature 0.4. 0 is a natural number.

Let zero stand for 0. Let n is nonzero stand for $n \neq 0$.

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Signature 0.5. 1 is a natural number.

Let one stand for 1. Let the direct successor of n stand for $n + 1$.

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Definition 0.6. $2 = 1 + 1$.

Let two stand for 2.

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Definition 0.7. $3 = 2 + 1$.

Let three stand for 3.

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Definition 0.8. $4 = 3 + 1$.

Let four stand for 4.

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Definition 0.9. $5 = 4 + 1$.

Let five stand for 5.

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Definition 0.10. $6 = 5 + 1$.

Let six stand for 6.

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Definition 0.11. $7 = 6 + 1$.

Let seven stand for 7.

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Definition 0.12. $8 = 7 + 1$.

Let eight stand for 8.

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Definition 0.13. $9 = 8 + 1$.

Let nine stand for 9.

The Axioms of Natural Number Arithmetic

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Axiom 0.14. Let n, m be natural numbers. If $n + 1 = m + 1$ then $n = m$.

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Axiom 0.15. There exists no natural number n such that $n + 1 = 0$.

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Axiom 0.16 (Induction). Let Φ be a class. Assume $0 \in \Phi$ and for all natural numbers n if $n \in \Phi$ then $n+1 \in \Phi$. Then Φ contains every natural number.

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Axiom 0.17. Then $1 = 0 + 1$.

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Axiom 0.18. Let n be a natural number. Then $n + 0 = n$.

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Axiom 0.19. Let n, m be natural numbers. Then $n + (m + 1) = (n + m) + 1$.

Immediate Consequences of the Axioms

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Proposition 0.20. Let n be a natural number. Then $n = 0$ or $n = m + 1$ for some natural number m.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid n' = 0 \text{ or } n' = m' + 1 \text{ for some natural number } m'\}$. $0 \in \Phi$ and for all $n' \in \Phi$ we have $n' + 1 \in \Phi$. Hence every natural number is contained in Φ. Thus $n = 0$ or $n = m + 1$ for some natural number m. $\square$

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Proposition 0.21. Let n be a natural number. Then $n \neq n + 1$.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid n' \neq n' + 1\}$.

(1) 0 belongs to Φ.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.
Proof. Let $n' \in \Phi$. Then $n' \neq n' + 1$. If $n' + 1 = (n' + 1) + 1$ then $n' = n' + 1$. Thus it is wrong that $n' + 1 = (n' + 1) + 1$. Hence $n' + 1 \in \Phi$. Qed.

Therefore every natural number is an element of Φ. Consequently $n \neq n+1$. □

Computation Laws for Addition

Associativity

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Proposition 0.22. Let n, m, k be natural numbers. Then $n + (m + k) = (n + m) + k$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid n + (m + k') = (n + m) + k'\}$.

(1) 0 is contained in Φ. Indeed $n + (m + 0) = n + m = (n + m) + 0$.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Then $n + (m + k') = (n + m) + k'$. Hence

$$\begin{aligned} & n + (m + (k' + 1)) \\ &= n + ((m + k') + 1) \\ &= (n + (m + k')) + 1 \\ &= ((n + m) + k') + 1 \\ &= (n + m) + (k' + 1). \end{aligned}$$

Thus $k' + 1 \in \Phi$. Qed.

Thus every natural number is an element of Φ. Therefore $n + (m + k) = (n + m) + k$. □

Commutativity

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Proposition 0.23. Let n, m be natural numbers. Then $n+m = m+n$.

Proof. Define $\Phi = \{m' \in \mathbb{N} \mid n + m' = m' + n\}$.

(1) 0 is an element of Φ.
Proof. Define $\Psi = \{n' \in \mathbb{N} \mid n' + 0 = 0 + n'\}$.

(1a) 0 belongs to Ψ.

(1b) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.
Proof. Let $n' \in \Psi$. Then $n' + 0 = 0 + n'$. Hence

$$(n' + 1) + 0$$

$$= n' + 1$$

$$= (n' + 0) + 1$$

$$= (0 + n') + 1$$

$$= 0 + (n' + 1).$$

Qed.

Hence every natural number belongs to Ψ. Thus $n + 0 = 0 + n$. Therefore 0 is an element of Φ. Qed.

Let us show that (2) $n + 1 = 1 + n$.
Proof. Define $\Theta = \{n' \in \mathbb{N} \mid n' + 1 = 1 + n'\}$.

(2a) 0 is an element of Θ.

(2b) For all $n' \in \Theta$ we have $n' + 1 \in \Theta$.
Proof. Let $n' \in \Theta$. Then $n' + 1 = 1 + n'$. Hence

$$(n' + 1) + 1$$

$$= (1 + n') + 1$$

$$= 1 + (n' + 1).$$

Thus $n' + 1 \in \Theta$. Qed.

Thus every natural number belongs to Θ. Therefore $n + 1 = 1 + n$. Qed.

(3) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.
Proof. Let $m' \in \Phi$. Then $n + m' = m' + n$. Hence

$$n + (m' + 1)$$

$$= (n + m') + 1$$

$$= (m' + n) + 1$$

$$= m' + (n + 1)$$

$$= m' + (1 + n)$$

$$= (m' + 1) + n.$$

Thus $m' + 1 \in \Phi$. Qed.

Thus every natural number is an element of Φ. Therefore $n + m = m + n$. ☐

Cancellation

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Proposition 0.24. Let n, m, k be natural numbers. If $n + k = m + k$ then $n = m$.

Proof. Define $\Phi = \{k' \in \mathbb{N} \mid \text{if } n + k' = m + k' \text{ then } n = m\}$.

(1) 0 is an element of Φ.

(2) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.
Proof. Let $k' \in \Phi$. Suppose $n + (k' + 1) = m + (k' + 1)$. Then $(n + k') + 1 = (m + k') + 1$. Hence $n + k' = m + k'$. Thus $n = m$. Qed.

Therefore every natural number is an element of Φ. Consequently if $n + k = m + k$ then $n = m$. ☐

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Corollary 0.25. Let n, m, k be natural numbers. If $k + n = k + m$ then $n = m$.

Proof. Assume $k + n = k + m$. We have $k + n = n + k$ and $k + m = m + k$. Hence $n + k = m + k$. Thus $n = m$. ☐

Zero Sums

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Proposition 0.26. Let n, m be natural numbers. If $n + m = 0$ then $n = 0$ and $m = 0$.

Proof. Assume $n + m = 0$. Suppose $n \neq 0$ or $m \neq 0$. Then we can take a $k \in \mathbb{N}$ such that $n = k + 1$ or $m = k + 1$. Hence there exists a natural number l such that $n + m = l + (k + 1) = (l + k) + 1 \neq 0$. Contradiction. ☐