Successors

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Definition 0.1. Let α be an ordinal. $\mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$.

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Proposition 0.2. Let α be an ordinal. Then $\mathrm{succ}(\alpha)$ is an ordinal.

Proof. (1) $\mathrm{succ}(\alpha)$ is transitive.
Proof. Let $x \in \mathrm{succ}(\alpha)$ and $y \in x$. Then $x \in \alpha$ or $x = \alpha$. Hence $y \in \alpha$. Thus $y \in \mathrm{succ}(\alpha)$. Qed.

(2) Every element of $\mathrm{succ}(\alpha)$ is transitive.
Proof. Let $x \in \mathrm{succ}(\alpha)$. Then $x \in \alpha$ or $x = \alpha$. Hence x is transitive. Indeed α is transitive and every element of α is transitive. Qed. □

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Definition 0.3. A successor ordinal is an ordinal α such that $\alpha = \mathrm{succ}(\beta)$ for some ordinal β.

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Proposition 0.4. Let α, β be ordinals. If $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$ then $\alpha = \beta$.

Proof. Assume $\mathrm{succ}(\alpha) = \mathrm{succ}(\beta)$.

(1) $\alpha \subseteq \beta$.
Proof. Let $\gamma \in \alpha$. Then $\gamma \in \alpha \cup \{\alpha\} = \mathrm{succ}(\alpha) = \mathrm{succ}(\beta) = \beta \cup \{\beta\}$. Hence $\gamma \in \beta$ or $\gamma = \beta$. Assume $\gamma = \beta$. Then $\beta \in \alpha$. Hence $\beta = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\alpha \setminus \{\gamma\}) \cup \{\alpha\}$. Therefore $\alpha \in \beta$. Consequently $\alpha \in \beta \in \alpha$. Contradiction. Qed.

(2) $\beta \subseteq \alpha$.
Proof. Let $\gamma \in \beta$. Then $\gamma \in \beta \cup \{\beta\} = \mathrm{succ}(\beta) = \mathrm{succ}(\alpha) = \alpha \cup \{\alpha\}$. Hence $\gamma \in \alpha$ or $\gamma = \alpha$. Assume $\gamma = \alpha$. Then $\alpha \in \beta$. Hence $\alpha = (\alpha \cup \{\alpha\}) \setminus \{\gamma\} = (\beta \cup \{\beta\}) \setminus \{\gamma\} = (\beta \setminus \{\gamma\}) \cup \{\beta\}$. Therefore $\beta \in \alpha$. Consequently $\beta \in \alpha \in \beta$. Contradiction. Qed. □

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Definition 0.5. Let α be a successor ordinal. $\mathrm{pred}(\alpha)$ is the ordinal β such that $\alpha = \mathrm{succ}(\beta)$.