

## Cardinal Numbers

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**Definition 0.1.** Let  $x$  be a set. The cardinality of  $x$  is the ordinal  $\kappa$  such that  $\kappa$  is equinumerous to  $x$  and every ordinal that is equinumerous to  $x$  is greater than or equal to  $\kappa$ .

Let  $|x|$  stand for the cardinality of  $x$ .

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**Definition 0.2.** A cardinal number is an ordinal  $\kappa$  such that  $\kappa = |x|$  for some set  $x$ .

Let a cardinal stand for a cardinal number.

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**Proposition 0.3.** Let  $\kappa$  be a cardinal. Then  $|\kappa| = \kappa$ .

*Proof.*  $\kappa$  is an ordinal that is equinumerous to  $\kappa$ . Hence  $|\kappa| \leq \kappa$ . Consider a set  $x$  such that  $\kappa = |x|$ . Then  $|\kappa|$  is an ordinal that is equinumerous to  $x$ . Hence  $\kappa \leq |\kappa|$ . Thus  $|\kappa| = \kappa$ .  $\square$

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**Proposition 0.4.** Let  $x, y$  be sets. Then  $x$  and  $y$  are equinumerous iff  $|x| = |y|$ .

*Proof.* Case  $x$  and  $y$  are equinumerous. Take a bijection  $f$  between  $x$  and  $y$ . Consider a bijection  $g$  between  $y$  and  $|y|$ . Then  $g \circ f$  is a bijection between  $x$  and  $|y|$  (by FOUNDATIONS\_08\_6435206693126144). Hence  $x$  and  $|y|$  are equinumerous. Thus  $|y| \geq |x|$ .

$f^{-1}$  is a bijection between  $y$  and  $x$ . Consider a bijection  $h$  between  $x$  and  $|x|$ . Then  $h \circ f^{-1}$  is a bijection between  $y$  and  $|x|$  (by FOUNDATIONS\_08\_6435206693126144). Hence  $y$  and  $|x|$  are equinumerous. Thus  $|x| \geq |y|$ .

Therefore  $|x| = |y|$ . End.

Case  $|x| = |y|$ . Consider a bijection  $f$  between  $x$  and  $|x|$  and a bijection  $g$  between  $|y|$  and  $y$ . Then  $g \circ f$  is a bijection between  $x$  and  $y$ . Hence  $x$  and  $y$  are equinumerous. End.  $\square$