Even and Odd Numbers

Definition

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Definition 0.1. Let n be a natural number. n is even iff n is divisible by 2.

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Definition 0.2. Let n be a natural number. n is odd iff n is not divisible by 2.

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Proposition 0.3. Let n be a natural number. n is even iff $n = 2 \cdot m$ for some natural number m.

Proof. Case n is even. Then 2 divides n. Hence $n = 2 \cdot m$ for some natural number m. End.

Case $n = 2 \cdot m$ for some natural number m. Then 2 divides n. Hence 2 is even. End. □

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Proposition 0.4. Let n be a natural number. n is odd iff $n = (2 \cdot m) + 1$ for some natural number m.

Proof. Case n is odd. (a) Define $\Phi = \{n' \in \mathbb{N} \mid \text{if } n' \text{ is odd then } n' = (2 \cdot m) + 1 \text{ for some natural number } m\}$.

(1) Φ contains 0. Indeed 0 is not odd.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.
Proof. Let $n' \in \Phi$.

Let us show that if $n' + 1$ is odd then $(n' + 1) = (2 \cdot m) + 1$ for some natural number m. Assume that $n' + 1$ is odd.

Case n' is even. Take a natural number m such that $n' = 2 \cdot m$. Then $n' + 1 = (2 \cdot m) + 1$. End.

Case n' is odd. Take a natural number m such that $n' = (2 \cdot m) + 1$. Then $n' + 1 = ((2 \cdot m) + 1) + 1 = (2 \cdot m) + (1 + 1) = (2 \cdot m) + 2 = 2 \cdot (m + 1)$. Hence 2 divides n'. Thus n' is even. Contradiction. End. End. Qed.

Then Φ contains every natural number (by `ARITHMETIC_01_4764664342`

773760). Hence n is odd iff $n = (2 \cdot m) + 1$ for some natural number m (by a). End.

Case $n = (2 \cdot m) + 1$ for some natural number m. Consider a natural number m such that $n = (2 \cdot m) + 1$. Assume that n is even. Then we can take a natural number k such that $n = 2 \cdot k$. Then we have $2 \cdot k = (2 \cdot m) + 1$. Hence 2 divides $(2 \cdot m) + 1$. Thus 2 divides 1. Indeed 2 divides $2 \cdot m$. Contradiction. End. □

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Proposition 0.5. Let n be a natural number. n is odd iff $n = (2 \cdot m) - 1$ for some positive natural number m.

Proof. Case n is odd. Consider a natural number k such that $n = (2 \cdot k) + 1$. Take $m = k + 1$. Then $n = (2 \cdot k) + 1 = (2 \cdot (k + 0)) + 1 = (2 \cdot (k + (1 - 1))) + 1 = (2 \cdot ((k + 1) - 1)) + 1 = (2 \cdot (m - 1)) + 1 = ((2 \cdot m) - (2 \cdot 1)) + 1 = ((2 \cdot m) - 2) + 1 = (2 \cdot m) - 1$. End.

Case $n = (2 \cdot m) - 1$ for some positive natural number m. Consider a positive natural number m such that $n = (2 \cdot m) - 1$. Take $k = m - 1$. Then $n = (2 \cdot m) - 1 = (2 \cdot (m + 0)) - 1 = (2 \cdot (m + (1 - 1))) - 1 = (2 \cdot ((m + 1) - 1)) - 1 = (2 \cdot ((m - 1) + 1)) - 1 = (2 \cdot (k + 1)) - 1 = ((2 \cdot k) + (2 \cdot 1)) - 1 = ((2 \cdot k) + 2) - 1 = (2 \cdot k) + (2 - 1) = (2 \cdot k) + 1$. Hence n is odd. End. □

Addition of Even and Odd Numbers

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Proposition 0.6. Let n, m be natural numbers. Assume that n and m are even. Then $n + m$ is even.

Proof. Take natural numbers k, l such that $n = 2 \cdot k$ and $m = 2 \cdot l$. Then $n + m = (2 \cdot k) + (2 \cdot l) = 2 \cdot (k + l)$. Hence $n + m$ is even. □

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Proposition 0.7. Let n, m be natural numbers. Assume that n is even and m is odd. Then $n + m$ is odd.

Proof. Take natural numbers k, l such that $n = 2 \cdot k$ and $m = (2 \cdot l) + 1$. Then $n + m = (2 \cdot k) + ((2 \cdot l) + 1) = ((2 \cdot k) + (2 \cdot l)) + 1 = (2 \cdot (k + l)) + 1$. Hence $n + m$ is odd. □

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Corollary 0.8. Let n, m be natural numbers. Assume that n is odd and m is even. Then $n + m$ is odd.

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Proposition 0.9. Let n, m be natural numbers. Assume that n and m are odd. Then $n + m$ is even.

Proof. Take natural numbers k, l such that $n = (2 \cdot k) + 1$ and $m = (2 \cdot l) + 1$. Then $n + m = ((2 \cdot k) + 1) + ((2 \cdot l) + 1) = (((2 \cdot k) + 1) + (2 \cdot l)) + 1 = ((2 \cdot k) + (1 + (2 \cdot l))) + 1 = ((2 \cdot k) + ((2 \cdot l) + 1)) + 1 = (((2 \cdot k) + (2 \cdot l)) + 1) + 1 = ((2 \cdot k) + (2 \cdot l)) + (1 + 1) = ((2 \cdot k) + (2 \cdot l)) + 2 = (2 \cdot (k + l)) + 2 = 2 \cdot ((k + l) + 1)$. Hence $n + m$ is even. □

0.1 Subtraction of even and odd numbers

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Proposition 0.10. Let n, m be natural numbers such that $n \geq m$. Assume that n, m are even. Then $n - m$ is even.

Proof. Take natural numbers k, l such that $n = 2 \cdot k$ and $m = 2 \cdot l$. Then $k \geq l$. Hence $n - m = (2 \cdot k) - (2 \cdot l) = 2 \cdot (k - l)$. Thus $n - m$ is even. □

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Proposition 0.11. Let n, m be natural numbers such that $n \geq m$. Assume that n is even and m is odd. Then $n - m$ is odd.

Proof. Take natural numbers k, l such that $n = 2 \cdot k$ and $m = (2 \cdot l) + 1$. Then $k \geq l$ and $2 \cdot (k - l) \geq 1$. Hence $n - m = (2 \cdot k) - ((2 \cdot l) + 1) = ((2 \cdot k) - (2 \cdot l)) - 1 = (2 \cdot (k - l)) - 1$. Thus $n - m$ is odd. □

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Corollary 0.12. Let n, m be natural numbers such that $n \geq m$. Assume that n is odd and m is even. Then $n - m$ is odd.

Proof. Take natural numbers k, l such that $n = (2 \cdot k) + 1$ and $m = 2 \cdot l$. Then $k \geq l$. Hence $n - m = ((2 \cdot k) + 1) - (2 \cdot l) = (1 + (2 \cdot k)) - (2 \cdot l) = 1 + ((2 \cdot k) - (2 \cdot l)) = ((2 \cdot k) - (2 \cdot l)) + 1 = (2 \cdot (k - l)) + 1$. Indeed $((2 \cdot k) - (2 \cdot l)) = 2 \cdot (k - l)$. Thus $n - m$ is odd. □

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Proposition 0.13. Let n, m be natural numbers such that $n \geq m$. Assume that n, m are odd. Then $n - m$ is even.

Proof. Take natural numbers k, l such that $n = (2 \cdot k) + 1$ and $m = (2 \cdot l) + 1$. Then $k \geq l$. Indeed $2 \cdot k \geq 2 \cdot l$. Hence $1 + (2 \cdot k) \geq 2 \cdot l$. Thus $n - m = ((2 \cdot k) + 1) - ((2 \cdot l) + 1) = ((1 + (2 \cdot k)) - (2 \cdot l)) - 1 = (1 + ((2 \cdot k) - (2 \cdot l))) - 1 = (1 + (2 \cdot (k - l))) - 1 = ((2 \cdot (k - l)) + 1) - 1 = 2 \cdot (k - l)$. Indeed $((2 \cdot k) - (2 \cdot l)) = 2 \cdot (k - l)$. Therefore $n - m$ is even. $\square$

Multiplication of Even and Odd Numbers

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Proposition 0.14. Let n, m be natural numbers. Assume that n is even or n is even. Then $n \cdot m$ is even.

Proof. Case n is even. Take a natural number k such that $n = 2 \cdot k$. Then $n \cdot m = (2 \cdot k) \cdot m = 2 \cdot (k \cdot m)$. Hence $n \cdot m$ is even. End.

Case m is even. Take a natural number l such that $m = 2 \cdot l$. Then $n \cdot m = n \cdot (2 \cdot l) = (n \cdot 2) \cdot l = (2 \cdot n) \cdot l = 2 \cdot (n \cdot l)$. Hence $n \cdot m$ is even. End. $\square$

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Proposition 0.15. Let n, m be natural numbers. Assume that n and m are odd. Then $n \cdot m$ is odd.

Proof. Take natural numbers k, l such that $n = (2 \cdot k) + 1$ and $m = (2 \cdot l) + 1$. Then $n \cdot m = ((2 \cdot k) + 1) \cdot m = ((2 \cdot k) \cdot m) + (1 \cdot m) = ((2 \cdot k) \cdot m) + m = (2 \cdot (k \cdot m)) + m$. $2 \cdot (k \cdot m)$ is even and m is odd. Hence $(2 \cdot (k \cdot m)) + m$ is odd. Therefore $n \cdot m$ is odd. $\square$