

## Closure Under Finite Intersections

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**Definition 0.1.** Let  $X$  be a system of sets.  $X$  is closed under finite intersections iff  $\bigcap U \in X$  for every nonempty finite subclass  $U$  of  $X$ .

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**Proposition 0.2.** Let  $X$  be a system of sets.  $X$  is closed under finite intersections iff  $U \cap V \in X$  for every  $U, V \in X$ .

*Proof.* Case  $X$  is closed under finite intersections. Let  $U, V \in X$ . Then  $\{U, V\}$  is a nonempty finite subclass of  $X$ . Hence  $U \cap V = \bigcap\{U, V\} \in X$ . End.

Case  $U \cap V \in X$  for every  $U, V \in X$ . Define  $\Phi = \{n \in \mathbb{N} \mid \bigcap U \in X \text{ for every nonempty subclass } U \text{ of } X \text{ that has } n \text{ elements}\}$ .

(1)  $\Phi$  contains 0.

(2) For every  $n \in \Phi$  we have  $n + 1 \in \Phi$ .

*Proof.* Let  $n \in \Phi$ . Then  $\bigcap U \in X$  for every nonempty subclass  $U$  of  $X$  that has  $n$  elements.

Let us show that  $\bigcap U \in X$  for every nonempty subclass  $U$  of  $X$  that has  $n + 1$  elements.

Case  $n = 0$ . Obvious.

Case  $n \neq 0$ . Let  $U$  be a nonempty subclass of  $X$  such that  $U$  has  $n + 1$  elements. Take a bijection  $f$  between  $\{1, \dots, n + 1\}$  and  $U$ . We have  $\{1, \dots, n + 1\} = \{1, \dots, n\} \cup \{n + 1\}$ . Take  $V = f[\{1, \dots, n\}]$ . We have  $\{1, \dots, n\} \subseteq \{1, \dots, n + 1\}$ .

Let us show that  $V \subseteq U$ . Let  $x \in V$ . Take  $k \in \{1, \dots, n\}$  such that  $x = f(k)$ . Indeed we can show that there exists a  $k \in \{1, \dots, n\}$  such that  $x = f(k)$ . Assume the contrary. Then  $x \neq f(k)$  for all  $k \in \{1, \dots, n\}$ . Hence  $x \notin f[\{1, \dots, n\}] = V$ . Contradiction. End. Hence  $x \in U$ . Indeed  $x \in f[\{1, \dots, n + 1\}]$ . End.

$V$  is nonempty. Indeed  $f(1) \in f[\{1, \dots, n\}]$ . Indeed  $1 \in \{1, \dots, n\}$ . Hence  $f \upharpoonright \{1, \dots, n\}$  is a bijection between  $\{1, \dots, n\}$  and  $V$  (by ??). Thus  $V$  has  $n$  elements. Consequently  $\bigcap V \in X$ .

Let us show that  $U = V \cup \{f(n + 1)\}$ .

(1)  $f[A \cup B] = f[A] \cup f[B]$  for all  $A, B \subseteq \text{dom}(f)$ .

(2)  $f[\{a\}] = \{f(a)\}$  for all  $a \in \text{dom}(f)$ .

Hence  $U = f[\text{dom}(f)] = f[\{1, \dots, n + 1\}] = f[\{1, \dots, n\} \cup \{n + 1\}] =$

$f[\{1, \dots, n\}] \cup f[\{n+1\}] = f[\{1, \dots, n\}] \cup \{f(n+1)\} = V \cup \{f(n+1)\}$ .  
Indeed  $n+1 \in \text{dom}(f)$  and  $\{1, \dots, n\}, \{1+n\} \subseteq \text{dom}(f)$ . End.

Let us show that  $\bigcap(A \cup B) = (\bigcap A) \cap (\bigcap B)$  for any nonempty systems of sets  $A, B$ . Let  $A, B$  be nonempty systems of sets.  $\bigcap(A \cup B) \subseteq (\bigcap A) \cap (\bigcap B)$ .  
 $((\bigcap A) \cap (\bigcap B)) \subseteq \bigcap(A \cup B)$ . End.

$f(n+1)$  and  $f(k)$  are sets for every  $k \in \{1, \dots, n\}$ . Indeed  $f(k) \in U$  for every  $k \in \{1, \dots, n\}$ . Hence  $V$  and  $\{f(n+1)\}$  are nonempty systems of sets. Thus  $\bigcap U = \bigcap(V \cup \{f(n+1)\}) = (\bigcap V) \cap (\bigcap \{f(n+1)\}) = (\bigcap V) \cap f(n+1) \in X$ . End. End. Qed.

Therefore  $\Phi$  contains every natural number (by ARITHMETIC\_01\_476466 4342773760). Thus  $\bigcap U \in X$  for every nonempty finite subclass  $U$  of  $X$ . Consequently  $X$  is closed under finite intersections. End.  $\square$