

# The Knaster-Tarski fixed point theorem and the Cantor-Schröder-Bernstein Theorem

Naproche formalization:

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This is a formalization of the *Knaster-Tarski Fixed Point Theorem* (i.e. of the assertion that every subset-preserving map has a fixed point) and of the *Cantor-Schröder-Bernstein Theorem* (i.e. of the fact that two sets are equinumerous iff they can be embedded into each other), where the proof of the latter theorem is based on the former one, as in [Schroeder2012].

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[readtex libraries/source/foundations/equinumerosity.ftl.tex]
[readtex libraries/source/foundations/maps-and-systems-of-sets.
ftl.tex]
[readtex libraries/source/set-theory/zf.ftl.tex]
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## The Knaster-Tarski Fixed Point Theorem

**Theorem (Knaster-Tarski).** Let  $x$  be a set. Let  $h$  be a map from  $\mathcal{P}(x)$  to  $\mathcal{P}(x)$  that preserves subsets. Then  $h$  has a fixed point.

*Proof.* (1) Define  $A = \{y \mid y \subseteq x \text{ and } y \subseteq h(y)\}$ . Then  $A$  is a subset of  $\mathcal{P}(x)$ . We have  $\bigcup A \in \mathcal{P}(x)$ .

Let us show that (2)  $\bigcup A \subseteq h(\bigcup A)$ . Let  $u \in \bigcup A$ . Take  $y \in A$  such that  $u \in y$ . Then  $u \in h(y)$ . We have  $y \subseteq \bigcup A$ . Hence  $h(y) \subseteq h(\bigcup A)$ . Thus  $h(y) \subseteq h(\bigcup A)$ . Therefore  $u \in h(\bigcup A)$ . End.

Then  $h(\bigcup A) \in A$  (by 1). Indeed  $h(\bigcup A) \subseteq x$ . (3) Hence  $h(\bigcup A) \subseteq \bigcup A$ . Indeed every element of  $h(\bigcup A)$  is an element of some element of  $A$ .

Thus  $h(\bigcup A) = \bigcup A$  (by 2, 3).  $\square$

## The Cantor-Schröder-Bernstein Theorem

**Theorem (Cantor-Schröder-Bernstein).** Let  $x, y$  be sets.  $x$  and  $y$  are equinumerous iff there exists a injective map from  $x$  to  $y$  and there exists an injective map from  $y$  to  $x$ .

*Proof.* Case  $x$  and  $y$  are equinumerous. Take a bijection  $f$  between  $x$  and  $y$ . Then  $f^{-1}$  is a bijection between  $y$  and  $x$ . Hence  $f$  is an injective map from  $x$  to  $y$  and  $f^{-1}$  is an injective map from  $y$  to  $x$ . End.

Case there exists an injective map from  $x$  to  $y$  and there exists an injective map from  $y$  to  $x$ . Take an injective map  $f$  from  $x$  to  $y$ . Take an injective map  $g$  from  $y$  to  $x$ . We have  $y \setminus f[a] \subseteq y$  for any  $a \in \mathcal{P}(x)$ .

(1) Define  $h(a) = x \setminus g[y \setminus f[a]]$  for  $a \in \mathcal{P}(x)$ .

$h$  is a map from  $\mathcal{P}(x)$  to  $\mathcal{P}(x)$ . Indeed  $h(a)$  is a subset of  $x$  for each subset  $a$  of  $x$ .

Let us show that  $h$  preserves subsets. Let  $u, v$  be subsets of  $x$ . Assume  $u \subseteq v$ . Then  $f[u] \subseteq f[v]$ . Hence  $y \setminus f[v] \subseteq y \setminus f[u]$ . Thus  $g[y \setminus f[v]] \subseteq g[y \setminus f[u]]$ . Indeed  $y \setminus f[v]$  and  $y \setminus f[u]$  are subsets of  $y$ . Therefore  $x \setminus g[y \setminus f[u]] \subseteq x \setminus g[y \setminus f[v]]$ . Consequently  $h[u] \subseteq h[v]$ . End.

Hence we can take a fixed point  $c$  of  $h$  (by [Knaster-Tarski](#)).

(2) Define  $F(u) = f(u)$  for  $u \in c$ .

We have  $c = h(c)$  iff  $x \setminus c = g[y \setminus f[c]]$ .  $g^{-1}$  is a bijection between  $\text{range}(g)$  and  $y$ . Thus  $x \setminus c = g[y \setminus f[c]] \subseteq \text{range}(g)$ . Therefore  $x \setminus c$  is a subset of  $\text{dom}(g^{-1})$ .

(3) Define  $G(u) = g^{-1}(u)$  for  $u \in x \setminus c$ .

$F$  is a bijection between  $c$  and  $\text{range}(F)$ .  $G$  is a bijection between  $x \setminus c$  and  $\text{range}(G)$ .

Define

$$H(u) = \begin{cases} F(u) & : u \in c \\ G(u) & : u \notin c \end{cases}$$

for  $u \in x$ .

Let us show that  $H$  is a map to  $y$ .  $\text{dom}(H)$  is a set and every value of  $H$  is an object. Hence  $H$  is a map.

Let us show that every value of  $H$  lies in  $y$ . Let  $v$  be a value of  $H$ . Take  $u \in x$  such that  $H(u) = v$ . If  $u \in c$  then  $v = H(u) = F(u) = f(u) \in y$ . If  $u \notin c$  then  $v = H(u) = G(u) = g^{-1}(u) \in y$ . End. End.

(4)  $H$  is surjective onto  $y$ . Indeed we can show that every element of  $y$  is a value of  $H$ . Let  $v \in y$ .

Case  $v \in f[c]$ . Take  $u \in c$  such that  $f(u) = v$ . Then  $F(u) = v$ . End.

Case  $v \notin f[c]$ . Then  $v \in y \setminus f[c]$ . Hence  $g(v) \in g[y \setminus f[c]]$ . Thus  $g(v) \in$

$x \setminus h(c)$ . We have  $g(v) \in x \setminus c$ . Therefore we can take  $u \in x \setminus c$  such that  $G(u) = v$ . Then  $v = H(u)$ . End. End.

(5)  $H$  is injective. Indeed we can show that for all  $u, v \in x$  if  $u \neq v$  then  $H(u) \neq H(v)$ . Let  $u, v \in x$ . Assume  $u \neq v$ .

Case  $u, v \in c$ . Then  $H(u) = F(u)$  and  $H(v) = F(v)$ . We have  $F(u) \neq F(v)$ . Hence  $H(u) \neq H(v)$ . End.

Case  $u, v \notin c$ . Then  $H(u) = G(u)$  and  $H(v) = G(v)$ . We have  $G(u) \neq G(v)$ . Hence  $H(u) \neq H(v)$ . End.

Case  $u \in c$  and  $v \notin c$ . Then  $H(u) = F(u)$  and  $H(v) = G(v)$ . Hence  $v \in g[y \setminus F[c]]$ . We have  $G(v) = g^{-1}(v) \in y \setminus F[c]$ . Indeed  $v = g(v')$  for some  $v' \in y \setminus F[c]$ . Thus  $G(v) \neq F(u)$ . End.

Case  $u \notin c$  and  $v \in c$ . Then  $H(u) = G(u)$  and  $H(v) = F(v)$ . Hence  $u \in g[y \setminus F[c]]$ . We have  $G(u) = g^{-1}(u) \in y \setminus F[c]$ . Indeed  $u = g(u')$  for some  $u' \in y \setminus F[c]$ . Thus  $G(u) \neq F(v)$ . End. End.

Consequently  $H$  is a bijection between  $x$  and  $y$  (by 4, 5). Therefore  $x$  and  $y$  are equinumerous. End.  $\square$

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