

Exponentiation and Ordering

ARITHMETIC_09_3373702288769024

Proposition 0.1. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $n^k < m^k$.

Proof. Case $n < m$. Define $\Phi = \{k' \in \mathbb{N} \mid \text{if } k' > 1 \text{ then } n^{k'} < m^{k'}\}$.

(1) Φ contains 0.

(2) Φ contains 1.

(3) Φ contains 2.

Proof. Case $n = 0$ or $m = 0$. Obvious.

Case $n, m \neq 0$. Then $n \cdot n < n \cdot m < m \cdot m$. Hence $n^2 = n \cdot n < n \cdot m < m \cdot m = m^2$. End. Qed.

(4) For all $k' \in \Phi$ we have $k' + 1 \in \Phi$.

Proof. Let $k' \in \Phi$.

Let us show that if $k' + 1 > 1$ then $n^{k'+1} < m^{k'+1}$. Assume $k' + 1 > 1$. Then $n^{k'} < m^{k'}$. Indeed $k' \neq 0$ and if $k' = 1$ then $n^{k'} < m^{k'}$.

Case $k' \leq 1$. Then $k' = 0$ or $k' = 1$. Hence $k' + 1 = 1$ or $k' + 1 = 2$. Thus $k' + 1 \in \Phi$. Therefore $n^{k'+1} < m^{k'+1}$. End.

Case $k' > 1$. Case $n = 0$. Then $m \neq 0$. Hence $n^{k'+1} = 0 < m^{k'} \cdot m = m^{k'+1}$. Thus $n^{k'+1} < m^{k'+1}$. End.

Case $n \neq 0$. Then $n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m$. Indeed $n^{k'} < m^{k'} \neq 0$. Take $A = n^{k'+1}$ and $B = m^{k'+1}$. Then $A = n^{k'+1} = n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m = m^{k'+1} = B$. Take $X = m^{k'} \cdot n$ and $Y = m^{k'} \cdot m$. Then $A < X < Y = B$. Hence $A < B$. Thus $n^{k'+1} < m^{k'+1}$. End. End.

Hence $n^{k'+1} < m^{k'+1}$. Indeed $k' \leq 1$ or $k' > 1$. End.

Thus $k' + 1 \in \Phi$. Qed.

Therefore every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Consequently $n^k < m^k$. End.

Case $n^k < m^k$. Define $\Psi = \{k' \in \mathbb{N} \mid \text{if } n \geq m \text{ then } n^{k'} \geq m^{k'}\}$.

(1) Ψ contains 0.

(2) For all $k' \in \Psi$ we have $k' + 1 \in \Psi$.

Proof. Let $k' \in \Psi$.

Let us show that if $n \geq m$ then $n^{k'+1} \geq m^{k'+1}$. Assume $n \geq m$. Then $n^{k'} \geq m^{k'}$. Hence $n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m$. Take $A = n^{k'+1}$ and $B = m^{k'+1}$. Thus $A = n^{k'+1} = n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m = m^{k'+1} = B$. Therefore $n^{k'+1} = A \geq B = m^{k'+1}$. End.

Hence $k' + 1 \in \Psi$. Qed.

Thus every natural number is contained in Ψ (by ARITHMETIC_01_476466 4342773760). Therefore if $n \geq m$ then $n^k \geq m^k$. Consequently $n < m$. End. \square

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Corollary 0.2. Let n, m, k be natural numbers. Assume $k \neq 0$. Then

$$n^k = m^k \quad \text{implies} \quad n = m.$$

Proof. Assume $n^k = m^k$. Suppose $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $n^k < m^k$. If $m < n$ then $m^k < n^k$. Thus $n^k \neq m^k$. Contradiction. \square

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Corollary 0.3. Let n, m, k be natural numbers. Assume $k \neq 0$. Then

$$n^k \leq m^k \quad \text{iff} \quad n \leq m.$$

Proof. If $n^k < m^k$ then $n < m$. If $n^k = m^k$ then $n = m$.
If $n < m$ then $n^k < m^k$. If $n = m$ then $n^k = m^k$. \square

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Proposition 0.4. Let n, m, k be natural numbers. Assume $k > 1$. Then

$$n < m \quad \text{iff} \quad k^n < k^m.$$

Proof. Case $n < m$. Define $\Phi = \{m' \in \mathbb{N} \mid \text{if } n < m' \text{ then } k^n < k^{m'}\}$.

(1) Φ contains 0.

(2) For all $m' \in \Phi$ we have $m' + 1 \in \Phi$.

Proof. Let $m' \in \Phi$.

Let us show that if $n < m' + 1$ then $k^n < k^{m'+1}$. Assume $n < m' + 1$. Then $n \leq m'$. We have $k^{m'} \cdot 1 < k^{m'} \cdot k$. Indeed $k^{m'} \neq 0$.

Case $n = m'$. Take $A = k^n$ and $B = k^{m'+1}$. Then $A = k^n = k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$. Hence $k^n = A < B = k^{m'+1}$. End.

Case $n < m'$. Take $A = k^n$ and $B = k^{m'+1}$. Then $A = k^n < k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$. Hence $k^n = A < B = k^{m'+1}$. End. Qed. Qed.

Hence every natural number is contained in Φ (by ARITHMETIC_01_4764 664342773760). Thus $k^n < k^m$. End.

Case $k^n < k^m$. Define $\Psi = \{n' \in \mathbb{N} \mid \text{if } n' \geq m \text{ then } k^{n'} \geq k^m\}$.

(1) 0 is contained in Ψ .

(2) For all $n' \in \Psi$ we have $n' + 1 \in \Psi$.

Proof. Let $n' \in \Psi$.

Let us show that if $n' + 1 \geq m$ then $k^{n'+1} \geq k^m$. Assume $n' + 1 \geq m$.

Case $n' + 1 = m$. Obvious.

Case $n' + 1 > m$. Then $n' \geq m$. Hence $k^{n'} \geq k^m$. We have $k^{n'} \cdot 1 \leq k^{n'} \cdot k$. Indeed $1 \leq k$ and $k^{n'} \neq 0$. Take $A = k^m$ and $B = k^{n'+1}$. Then $A = k^m \leq k^{n'} = k^{n'} \cdot 1 \leq k^{n'} \cdot k = k^{n'+1} = B$. Hence $k^m = A \leq B = k^{n'+1}$. End. Qed. Qed.

Thus every natural number is contained in Ψ (by ARITHMETIC_01_476466 4342773760). Therefore if $n \geq m$ then $k^n \geq k^m$. Consequently $n < m$. End. \square

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Corollary 0.5. Let n, m, k be natural numbers. Assume $k > 1$. If $k^n = k^m$ then $n = m$.

Proof. Assume $k^n = k^m$. Suppose $n \neq m$. Then $n < m$ or $m < n$. If $n < m$ then $k^n < k^m$. If $m < n$ then $k^m < k^n$. Thus $k^n \neq k^m$. Contradiction. \square

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Corollary 0.6. Let n, m, k be natural numbers. Assume $k > 1$. Then $n \leq m$ iff $k^n \leq k^m$.

Proof. If $n \leq m$ then $k^n \leq k^m$.

If $k^n = k^m$ then $n = m$. If $k^n < k^m$ then $n < m$. \square

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Proposition 0.7. Let n be a natural number. Then $(n + 1)^2 = (n^2 + (2 \cdot n)) + 1$.

Proof. We have

$$\begin{aligned} & (n + 1)^2 \\ &= (n + 1) \cdot (n + 1) \\ &= ((n + 1) \cdot n) + (n + 1) \\ &= ((n \cdot n) + n) + (n + 1) \end{aligned}$$

$$\begin{aligned}
&= (n^2 + n) + (n + 1) \\
&= ((n^2 + n) + n) + 1 \\
&= (n^2 + (n + n)) + 1 \\
&= (n^2 + (2 \cdot n)) + 1.
\end{aligned}$$

□

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Proposition 0.8. Let n be a natural number. Assume $n \geq 3$. Then $n^2 > (2 \cdot n) + 1$.

Proof. (a) Define $\Phi = \{n' \in \mathbb{N}_{\geq 3} \mid n'^2 > (2 \cdot n') + 1\}$.

(1) Φ contains 3. Indeed $3^2 > (2 \cdot 3) + 1$.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 3$.

Take $V = (n'^2 + (2 \cdot n')) + 1$ and $W = (((2 \cdot n') + 1) + (2 \cdot n')) + 1$. Then $V > W$.

Proof. We have $n'^2 > (2 \cdot n') + 1$. Hence $n'^2 + (2 \cdot n') > ((2 \cdot n') + 1) + (2 \cdot n')$ (by ARITHMETIC_04_7354062662008832). Qed.

Take $X = ((2 \cdot n') + (2 \cdot n')) + 1$. Then $W > X$.

Proof. We have $((2 \cdot n') + 1) + (2 \cdot n') > (2 \cdot n') + (2 \cdot n')$. Indeed $(2 \cdot n') + 1 > 2 \cdot n'$. Qed.

Take $Y = (2 \cdot (n' + n')) + 1$ and $Z = (2 \cdot (n' + 1)) + 1$. Then $Y > Z$.

Proof. We have $n' + n' > n' + 1$ and $2 \neq 0$. Thus $2 \cdot (n' + n') > 2 \cdot (n' + 1)$ (by ARITHMETIC_06_5048640368279552). Indeed $n' + n'$ and $n' + 1$ are natural numbers. Qed.

Then $(n' + 1)^2 = V > W > X = Y > Z = (2 \cdot (n' + 1)) + 1$. Hence $(n' + 1)^2 > (2 \cdot (n' + 1)) + 1$. Qed.

Therefore Φ contains every element of $\mathbb{N}_{\geq 3}$ (by ARITHMETIC_04_49765992 69113856, 1, 2). Consequently $n^2 > (2 \cdot n) + 1$ (by a). □

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Proposition 0.9. Let n be a natural number. Assume $n \geq 5$. Then $2^n > n^2$.

Proof. (a) Define $\Phi = \{n' \in \mathbb{N}_{\geq 5} \mid 2^{n'} > n'^2\}$.

(1) Φ contains 5. Indeed we can show that $2^5 > 5^2$. We have $2^5 = 2 \cdot (2 \cdot (2 \cdot (2 \cdot 2))) = 8 \cdot 4 = (5+3) \cdot 4 = (5 \cdot 4) + (3 \cdot 4) = (5 \cdot 4) + ((5 \cdot 2) + 2) = (5 \cdot (4+2)) + 2 =$

$(5 \cdot 6) + 2 > 5 \cdot 6 > 5 \cdot 5 = 5^2$. Indeed $((5 \cdot 4) + ((5 \cdot 2) + 2)) = (5 \cdot (4 + 2)) + 2$.
End.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 5$ and $2^{n'} > n'^2$.

Take $V = 2^{n'} \cdot 2$ and $W = n'^2 \cdot 2$. Then $V > W$. Indeed $2 \neq 0$.

Take $X = n'^2 + n'^2$. Then $W = X$.

Take $Y = n'^2 + ((2 \cdot n') + 1)$. Then $X > Y$. Indeed $n'^2 > (2 \cdot n') + 1$.

Take $Z = (n' + 1)^2$. Then $Y = Z$.

Then $2^{n'+1} = V > W = X > Y = Z = (n' + 1)^2$. Hence $2^{n'+1} > (n' + 1)^2$.
Qed.

Therefore Φ contains every element of $\mathbb{N}_{\geq 5}$ (by ARITHMETIC_04_49765992 69113856). Consequently $2^n > n^2$ (by a). \square