

## Natural Numbers are Cardinal Numbers

SET\_THEORY\_07\_2948332552978432

**Theorem 0.1.** Let  $n \in \omega$ . Then  $|n| = n$ .

*Proof.* Define  $\Phi = \{n' \in \omega \mid |n'| = n'\}$ .

(1)  $0 \in \Phi$ . Indeed  $|0| = |\emptyset| = 0$ .

(2) For all  $n' \in \Phi$  we have  $\text{succ}(n') \in \Phi$ .

*Proof.* Let  $n' \in \Phi$ . Then  $|n'| = n'$ . We have  $|\text{succ}(n')| \leq \text{succ}(n')$ .

Let us show that  $\text{succ}(n') \leq |\text{succ}(n')|$ . Assume the contrary. Then  $|\text{succ}(n')| < \text{succ}(n')$ . Take a bijection  $f$  between  $|\text{succ}(n')|$  and  $\text{succ}(n')$ .  $|\text{succ}(n')|$  is nonzero. Hence we can take a  $m \in \omega$  such that  $|\text{succ}(n')| = \text{succ}(m)$ . Indeed  $|\text{succ}(n')| \in \text{succ}(\text{succ}(n')) \in \omega$ . Then  $f^{-1}(n') \leq m$ .

We can show that  $f^{-1}(n') < m$ . Assume the contrary. Then  $f^{-1}(n') = m$ .  $f \upharpoonright m$  is a bijection between  $m$  and  $f[m]$  (by FOUNDATIONS\_08\_6474462 31252992). Indeed  $f$  is an injective map from  $|\text{succ}(n')|$  to  $\text{succ}(n')$  and  $m \subseteq |\text{succ}(n')|$ . We have  $f[m] \subseteq n'$  and  $n' \subseteq f[m]$ . Hence  $f[m] = n'$ . Thus  $f \upharpoonright m$  is a bijection between  $m$  and  $n'$ . Therefore  $n' = |n'| \leq m < |\text{succ}(n')| < \text{succ}(n')$ . Consequently  $m = n'$ . Then we have  $\text{succ}(n') = |\text{succ}(n')| < \text{succ}(n')$ . Contradiction. End.

Define

$$g(i) = \begin{cases} f(i) & : i \neq f^{-1}(n') \\ f(m) & : i = f^{-1}(n') \end{cases}$$

for  $i \in m$ .

$g$  is a map from  $m$  to  $n'$ . Indeed we can show that  $g(i) \in n'$  for each  $i \in m$ . *Proof.* Let  $i \in m$ . We have  $g(i) \in \text{succ}(n')$ . Indeed  $f(i), f(m) \in \text{succ}(n')$  and  $(g(i) = f(i) \text{ or } g(i) = f(m))$ . Hence if  $g(i) \neq n'$  then  $g(i) \in n'$ .

Case  $i \neq f^{-1}(n')$ . Then  $g(i) = f(i) \in \text{succ}(n')$ . If  $g(i) = n'$  then  $f(i) = n' = f(f^{-1}(n'))$ . Hence if  $g(i) = n'$  then  $i = f^{-1}(n')$ . Thus  $g(i) \neq n'$ . Therefore  $g(i) \in n'$ . End.

Case  $i = f^{-1}(n')$ . Then  $g(i) = f(m) \neq f(f^{-1}(n')) = n'$ . Hence  $g(i) \in n'$ . End. Qed.

$g$  is surjective onto  $n'$ . Indeed we can show that for all  $k \in n'$  there exists a  $l \in m$  such that  $k = g(l)$ .

*Proof.* Let  $k \in n'$ . Then  $f^{-1}(k) \neq f^{-1}(n')$ .

Case  $f^{-1}(k) = m$ . Then  $k = f(f^{-1}(k)) = f(m) = g(f^{-1}(n'))$ . End.

Case  $f^{-1}(k) \neq m$ . Then  $f^{-1}(k) \in m$ . Indeed  $f^{-1}(k) \in |\text{succ}(n')| = \text{succ}(m) = m \cup \{m\}$ . Hence  $k = f(f^{-1}(k)) = g(f^{-1}(k))$ . End. Qed.

$g$  is injective. Indeed we can show that for all  $i, j \in m$  if  $i \neq j$  then  $g(i) \neq g(j)$ .

Proof. Let  $i, j \in m$ . Assume  $i \neq j$ .

Case  $i, j \neq f^{-1}(n')$ . Then  $g(i) = f(i) \neq f(j) = g(j)$ . End.

Case  $i = f^{-1}(n')$ . Then  $j \neq f^{-1}(n')$ . Hence  $g(i) = g(f^{-1}(n')) = f(m) \neq f(j) = g(j)$ . Indeed  $m \neq j$ . End.

Case  $j = f^{-1}(n')$ . Then  $i \neq f^{-1}(n')$ . Hence  $g(i) = f(i) \neq f(m) = g(f^{-1}(n')) = g(j)$ . Indeed  $i \neq m$ . End. Qed. End. End.

Thus  $\omega \subseteq \Phi$  (by SET\_THEORY\_03\_344585425387520). Consequently  $n \in \Phi$ .  
Therefore  $|n| = n$ .  $\square$

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**Corollary 0.2.** Every natural number is a cardinal.