Multiplication and Ordering

ARITHMETIC_06_8817333933965312

Proposition 0.1. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $n \cdot k < m \cdot k$.

Proof. Case $n \cdot k < m \cdot k$. Define $\Phi = \{n' \in \mathbb{N} \mid \text{if } n' \cdot k < m \cdot k \text{ then } n' < m\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.
Proof. Let $n' \in \Phi$.

Let us show that if $(n'+1) \cdot k < m \cdot k$ then $n'+1 < m$. Assume $(n'+1) \cdot k < m \cdot k$. Then $(n' \cdot k) + k < m \cdot k$. Hence $n' \cdot k < m \cdot k$. Thus $n' < m$. Then $n' + 1 \leq m$. If $n' + 1 = m$ then $(n'+1) \cdot k = m \cdot k$. Hence $n' + 1 < m$. End. Qed.

Therefore every natural number is contained in Φ (by `ARITHMETIC_01_4764664342773760`). Consequently $n < m$. End.

Case $n < m$. Take a positive natural number l such that $m = n + l$. Then $m \cdot k = (n + l) \cdot k = (n \cdot k) + (l \cdot k)$. $l \cdot k$ is positive. Hence $n \cdot k < m \cdot k$. End. □

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Corollary 0.2. Let n, m, k be natural numbers. Assume $k \neq 0$. Then $n < m$ iff $k \cdot n < k \cdot m$.

Proof. The thesis (by `ARITHMETIC_06_8817333933965312`, `ARITHMETIC_06_1764759896588288`). □

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Proposition 0.3. Let n, m, k be natural numbers. If $n, m > k$ then $n \cdot m > k$.

Proof. Define $\Phi = \{n' \in \mathbb{N} \mid \text{if } n', m > k \text{ then } n' \cdot m > k\}$.

(1) Φ contains 0.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.
Proof. Let $n' \in \Phi$.

Let us show that if $n'+1, m > k$ then $(n'+1) \cdot m > k$. Assume $n'+1, m > k$. Then $(n'+1) \cdot m = (n' \cdot m) + m$. If $n' = 0$ then $(n' \cdot m) + m = 0 + m = m > k$. If $n' \neq 0$ then $(n' \cdot m) + m > m > k$. Indeed if $n' \neq 0$ then $n' \cdot m > 0$.

Indeed $m > 0$. Hence $(n' + 1) \cdot m > k$. Qed. Qed.

Thus every natural number is contained in Φ (by ARITHMETIC_01_4764664342773760). Therefore if $n, m > k$ then $n \cdot m > k$. □

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Corollary 0.4. Let n, m, k be natural numbers. If $n \leq m$ then $k \cdot n \leq k \cdot m$.

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Corollary 0.5. Let n, m, k be natural numbers. Assume $k \neq 0$. If $k \cdot n \leq k \cdot m$ then $n \leq m$.

Proof. If $k \cdot n = k \cdot m$ then $n = m$ (by ARITHMETIC_06_8575191374364672). If $k \cdot n < k \cdot m$ then $n < m$ (by ARITHMETIC_06_5048640368279552). □

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Corollary 0.6. Let n, m, k be natural numbers. If $n \leq m$ then $n \cdot k \leq m \cdot k$.

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Corollary 0.7. Let n, m, k be natural numbers. Assume $k \neq 0$. If $n \cdot k \leq m \cdot k$ then $n \leq m$.

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Proposition 0.8. Let n, m, k be natural numbers. Assume $m > 0$ and $k > 1$. Then $k \cdot m > m$.

Proof. Take a natural number l such that $k = l + 2$. Then

$$k \cdot m$$

$$= (l + 2) \cdot m$$

$$= (l \cdot m) + (2 \cdot m)$$

$$= (l \cdot m) + (m + m)$$

$$= ((l \cdot m) + m) + m$$

$$= ((l + 1) \cdot m) + m$$

$$\geq 1 + m$$
$$> m.$$

Indeed $((l+1) \cdot m) + m \geq 1 + m$. $\square$