Segments of the Natural Numbers

FOUNDATIONS_14_3625613501923328

Definition 0.1. Let n, m be natural numbers. $\{n, \ldots, m\} = \{k \in \mathbb{N} \mid n \leq k \leq m\}$.

FOUNDATIONS_14_145331933151232

Proposition 0.2. Let n, m be natural numbers. If $\{1, \ldots, n\} = \{1, \ldots, m\}$ then $n = m$.

Proof. Assume $\{1, \ldots, n\} = \{1, \ldots, m\}$.

Case $n = 0$. Then $\{1, \ldots, n\} = \emptyset$. Thus $\{1, \ldots, m\} = \emptyset$. Hence there exists no $k \in \mathbb{N}$ such that $1 \leq k \leq m$. Therefore $m = 0$. Consequently $n = m$. End.

Case $m = 0$. Then $\{1, \ldots, m\} = \emptyset$. Thus $\{1, \ldots, n\} = \emptyset$. Hence there exists no $k \in \mathbb{N}$ such that $1 \leq k \leq n$. Therefore $n = 0$. Consequently $n = m$. End.

Case $n, m \geq 1$. For all $k \in \mathbb{N}$ we have $1 \leq k \leq n$ iff $1 \leq k \leq m$. Hence for all $k \in \mathbb{N}$ we have $k \leq n$ iff $k \leq m$.

Let us show by contradiction that $n = m$. Suppose $n \neq m$. Then $n > m$ or $m > n$.

Case $n > m$. Take $k = m + 1$. Then $k \leq n$ and $k \nleq m$. Hence it is wrong that $k \leq n$ iff $k \leq m$. Contradiction. End.

Case $m > n$. Take $k = n + 1$. Then $k \leq m$ and $k \nleq m$. Hence it is wrong that $k \leq n$ iff $k \leq m$. Contradiction. End. End. End. $\square$

FOUNDATIONS_14_658708738605056

Proposition 0.3. Let n be a natural number. Then $\{1, \ldots, n+1\} = \{1, \ldots, n\} \cup \{n+1\}$.

Proof. We have $\{1, \ldots, n+1\} \subseteq \{1, \ldots, n\} \cup \{n+1\}$ and $\{1, \ldots, n\} \cup \{n+1\} \subseteq \{1, \ldots, n+1\}$. $\square$