Exponentiation and Divisibility

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Proposition 0.1. Let n, m, k be natural numbers such that $n, m \neq 0$ and $k > 1$. Then $k^n \mid k^m$ iff $n \leq m$.

Proof. Case $k^n \mid k^m$. Assume $n > m$. Take a nonzero natural number l such that $n = m + l$. Then $k^n = k^{m+l} = k^m \cdot k^l$. Hence $k^m \mid k^n$. Thus $k^m = k^n$. Therefore $m = n$. Contradiction. End.

Case $n \leq m$. Take a natural number l such that $m = n + l$. Then $k^m = k^{n+l} = k^n \cdot k^l$. Hence $k^n \mid k^m$. End. □

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Proposition 0.2. Let n be a composite natural number. Then n has a nontrivial divisor m such that $m^2 \leq n$.

Proof. Define $A = \{m \in \mathbb{N} \mid m \text{ is a nontrivial divisor of } n\}$. A is nonempty. Hence we can take a $m \in A$ such that $m \leq l$ for all $l \in A$. Consider a natural number k such that $m \cdot k = n$. Then $m \leq k$. Hence $m^2 = m \cdot m \leq m \cdot k = n$. Therefore $m^2 \leq n$. □