

# Exponentiation

## Definition

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**Signature 0.1.** Let  $n, m$  be natural numbers.  $n^m$  is a natural number.

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**Axiom 0.2.** Let  $n$  be a natural number. Then  $n^0 = 1$ .

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**Axiom 0.3.** Let  $n, m$  be natural numbers. Then  $n^{m+1} = n^m \cdot n$ .

## Computation Laws

### Exponentiation with 0, 1 and 2

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**Proposition 0.4.** Let  $n$  be a natural number. Assume  $n \neq 0$ . Then  $0^n = 0$ .

*Proof.* Take a natural number  $m$  such that  $n = m + 1$ . Then  $0^n = 0^{m+1} = 0^m \cdot 0 = 0$ . Indeed  $0^{m+1} = 0^m \cdot 0$ .  $\square$

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**Proposition 0.5.** Let  $n$  be a natural number. Then  $1^n = 1$ .

*Proof.* Define  $\Phi = \{n' \in \mathbb{N} \mid 1^{n'} = 1\}$ .

(1)  $\Phi$  contains 0.

(2) For all  $n' \in \Phi$  we have  $n' + 1 \in \Phi$ .

*Proof.* Let  $n' \in \Phi$ . Then  $1^{n'+1} = 1^{n'} \cdot 1 = 1 \cdot 1 = 1$ . Qed.

Hence every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_4764664342773760). Thus  $1^n = 1$ .  $\square$

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**Proposition 0.6.** Let  $n$  be a natural number. Then  $n^1 = n$ .

*Proof.* We have  $n^1 = n^{0+1} = n^0 \cdot n = 1 \cdot n = n$ . □

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**Proposition 0.7.** Let  $n$  be a natural number. Then  $n^2 = n \cdot n$ .

*Proof.* We have  $n^2 = n^{1+1} = n^1 \cdot n = n \cdot n$ . □

### Sums as Exponents

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**Proposition 0.8.** Let  $n, m, k$  be natural numbers. Then  $k^{n+m} = k^n \cdot k^m$ .

*Proof.* Define  $\Phi = \{m' \in \mathbb{N} \mid k^{n+m'} = k^n \cdot k^{m'}\}$ .

(1)  $\Phi$  contains 0.

Indeed  $k^{n+0} = k^n = k^n \cdot 1 = k^n \cdot k^0$ .

(2) For all  $m' \in \Phi$  we have  $m' + 1 \in \Phi$ .

*Proof.* Let  $m' \in \Phi$ . Then

$$\begin{aligned} & k^{n+(m'+1)} \\ &= k^{(n+m')+1} \\ &= k^{n+m'} \cdot k \\ &= (k^n \cdot k^{m'}) \cdot k \\ &= k^n \cdot (k^{m'} \cdot k) \\ &= k^n \cdot k^{m'+1}. \end{aligned}$$

Qed.

Hence every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_4764664342773760). Thus  $k^{n+m} = k^n \cdot k^m$ . □

### Products as Exponents

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**Proposition 0.9.** Let  $n, m, k$  be natural numbers. Then  $n^{m \cdot k} = (n^m)^k$ .

*Proof.* Define  $\Phi = \{k' \in \mathbb{N} \mid n^{m \cdot k'} = (n^m)^{k'}\}$ .

(1)  $\Phi$  contains 0. Indeed  $(n^m)^0 = 1 = n^0 = n^{m \cdot 0}$ .

(2) For all  $k' \in \Phi$  we have  $k' + 1 \in \Phi$ .

*Proof.* Let  $k' \in \Phi$ . Then

$$\begin{aligned} & (n^m)^{k'+1} \\ &= (n^m)^{k'} \cdot n^m \\ &= n^{m \cdot k'} \cdot n^m \\ &= n^{(m \cdot k') + m} \\ &= n^{m \cdot (k'+1)}. \end{aligned}$$

Qed.

Therefore every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_47 64664342773760). Consequently  $n^{m \cdot k} = (n^m)^k$ .  $\square$

## Products as Base

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**Proposition 0.10.** Let  $n, m, k$  be natural numbers. Then  $(n \cdot m)^k = n^k \cdot m^k$ .

*Proof.* Define  $\Phi = \{k' \in \mathbb{N} \mid (n \cdot m)^{k'} = n^{k'} \cdot m^{k'}\}$ .

(1)  $\Phi$  contains 0. Indeed  $((n \cdot m)^0) = 1 = 1 \cdot 1 = n^0 \cdot m^0$ .

(2) For all  $k' \in \Phi$  we have  $k' + 1 \in \Phi$ .

*Proof.* Let  $k' \in \Phi$ .

Let us show that  $(n^{k'} \cdot m^{k'}) \cdot (n \cdot m) = (n^{k'} \cdot n) \cdot (m^{k'} \cdot m)$ .

$$\begin{aligned} & (n^{k'} \cdot m^{k'}) \cdot (n \cdot m) \\ &= ((n^{k'} \cdot m^{k'}) \cdot n) \cdot m \\ &= (n^{k'} \cdot (m^{k'} \cdot n)) \cdot m \\ &= (n^{k'} \cdot (n \cdot m^{k'})) \cdot m \\ &= ((n^{k'} \cdot n) \cdot m^{k'}) \cdot m \\ &= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m). \end{aligned}$$

Qed.

Hence

$$(n \cdot m)^{k'+1}$$

$$\begin{aligned}
&= (n \cdot m)^{k'} \cdot (n \cdot m) \\
&= (n^{k'} \cdot m^{k'}) \cdot (n \cdot m) \\
&= (n^{k'} \cdot n) \cdot (m^{k'} \cdot m) \\
&= n^{k'+1} \cdot m^{k'+1}.
\end{aligned}$$

Qed.

Therefore every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_4764664342773760). Consequently  $(n \cdot m)^k = n^k \cdot m^k$ .  $\square$

## Zeroes of Exponentiation

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**Proposition 0.11.** Let  $n, m$  be natural numbers. Then  $n^m = 0$  iff  $n = 0$  and  $m \neq 0$ .

*Proof.* Case  $n^m = 0$ . Define  $\Phi = \{m' \in \mathbb{N} \mid \text{if } n^{m'} = 0 \text{ then } n = 0 \text{ and } m' \neq 0\}$ .

(1)  $\Phi$  contains 0. Indeed if  $n^0 = 0$  then we have a contradiction.

(2) For all  $m' \in \Phi$  we have  $m' + 1 \in \Phi$ .

*Proof.* Let  $m' \in \Phi$ .

Let us show that if  $n^{m'+1} = 0$  then  $n = 0$  and  $m' + 1 \neq 0$ . Assume  $n^{m'+1} = 0$ . Then  $0 = n^{m'+1} = n^{m'} \cdot n$ . Hence  $n^{m'} = 0$  or  $n = 0$ . We have  $m' + 1 \neq 0$  and if  $n^{m'} = 0$  then  $n = 0$ . Hence  $n = 0$  and  $m' + 1 \neq 0$ . End. Qed.

Thus every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_4764664342773760). Consequently  $m \in \Phi$ . Therefore  $n = 0$  and  $m \neq 0$ . End.

Case  $n = 0$  and  $m \neq 0$ . Take a natural number  $k$  such that  $m = k + 1$ . Then  $n^m = n^{k+1} = n^k \cdot n = 0^k \cdot 0 = 0$ . End.  $\square$