

# Dedekind's Recursion Theorem

Naproche formalization:

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2024

This is a formalization of Dedekind's recursion theorem. It states that for any function  $f : A \rightarrow A$  and any  $a \in A$  there exists a unique function  $\varphi : \mathbb{N} \rightarrow A$  that is *recursively defined* by  $f$  and  $a$ , i.e.  $\varphi(0) = a$  and  $\varphi(n+1) = f(\varphi(n))$ . (cf. [Ebbinghaus1991]).

**Definition.** Let  $a$  be an object and  $f$  be a map. Let  $\varphi$  be a map from  $\mathbb{N}$  to  $\text{dom}(f)$ .  $\varphi$  is recursively defined by  $a$  and  $f$  iff  $\varphi(0) = a$  and  $\varphi(n+1) = f(\varphi(n))$  for every  $n \in \mathbb{N}$ .

**Theorem (Dedekind's Recursion Theorem: Existence).** Let  $A$  be a set and  $a \in A$  and  $f : A \rightarrow A$ . Then there exists a  $\varphi : \mathbb{N} \rightarrow A$  that is recursively defined by  $a$  and  $f$ .

*Proof.* (a) Define

$$\Phi = \left\{ H \in \mathcal{P}(\mathbb{N} \times A) \mid \begin{array}{l} (0, a) \in H \text{ and for all } n \in \mathbb{N} \text{ and all } x \in A \text{ if} \\ (n, x) \in H \text{ then } (n+1, f(x)) \in H \end{array} \right\}.$$

Let us show that  $\bigcap \Phi \in \Phi$ .

Proof. (0)  $\mathbb{N} \times A \in \Phi$ .

Proof.

(i)  $\mathbb{N} \times A \in \mathcal{P}(\mathbb{N} \times A)$ .

(ii)  $(0, a) \in \mathbb{N} \times A$ .

(iii) for all  $n \in \mathbb{N}$  and all  $x \in A$  if  $(n, x) \in \mathbb{N} \times A$  then  $(n+1, f(x)) \in \mathbb{N} \times A$ .

Proof. Let  $n \in \mathbb{N}$  and  $x \in A$ . Assume  $(n, x) \in \mathbb{N} \times A$ . We have  $n+1 \in \mathbb{N}$  and  $f(x) \in A$ . Hence  $(n+1, f(x)) \in \mathbb{N} \times A$ . Qed. Qed.

(1)  $\bigcap \Phi \in \mathcal{P}(\mathbb{N} \times A)$ .

Proof. Any element of  $\bigcap \Phi$  is contained in every element of  $\Phi$ . Hence any element of  $\bigcap \Phi$  is contained in  $\mathbb{N} \times A$ . Thus  $\bigcap \Phi \subseteq \mathbb{N} \times A$ .  $\bigcap \Phi$  is a set. Hence  $\bigcap \Phi$  is a subset of  $\mathbb{N} \times A$ . Qed.

(2)  $(0, a) \in \bigcap \Phi$ .

Indeed  $(0, a) \in \mathbb{N} \times A \in \Phi$ .

(3) For all  $n \in \mathbb{N}$  and all  $x \in A$  if  $(n, x) \in \bigcap \Phi$  then  $(n+1, f(x)) \in \bigcap \Phi$ .

Proof. Let  $n \in \mathbb{N}$  and  $x \in A$ . Assume  $(n, x) \in \bigcap \Phi$ . Then  $(n, x)$  is contained in every element of  $\Phi$ . Hence  $(n+1, f(x))$  is contained in every element of  $\Phi$ . Thus  $(n+1, f(x)) \in \bigcap \Phi$ . Qed.

Therefore  $\bigcap \Phi \in \Phi$  (by a). Qed.

Let us show that for any  $n \in \mathbb{N}$  there exists an  $x \in A$  such that  $(n, x) \in \bigcap \Phi$ .

Proof. Define  $\Psi = \{n \in \mathbb{N} \mid \text{there exists an } x \in A \text{ such that } (n, x) \in \bigcap \Phi\}$ .

(1) 0 is contained in  $\Psi$ . Indeed  $(0, a) \in \bigcap \Phi$ .

(2) For all  $n \in \Psi$  we have  $n+1 \in \Psi$ .

Proof. Let  $n \in \Psi$ . Take an  $x \in A$  such that  $(n, x) \in \bigcap \Phi$ . Then  $(n+1, f(x)) \in \bigcap \Phi$ . Hence  $n+1 \in \Psi$ . Indeed  $f(x) \in A$ . Qed.

Therefore  $n \in \Psi$  for every  $n \in \mathbb{N}$  (by ARITHMETIC\_01\_4764664342773760). Qed.

Let us show that for all  $n \in \mathbb{N}$  and all  $x, y \in A$  if  $(n, x), (n, y) \in \bigcap \Phi$  then  $x = y$ .

Proof. (b) Define  $\Theta = \{n \in \mathbb{N} \mid \text{for all } x, y \in A \text{ if } (n, x), (n, y) \in \bigcap \Phi \text{ then } x = y\}$ .

(1)  $\Theta$  contains 0.

Proof. Let us show that for all  $x, y \in A$  if  $(0, x), (0, y) \in \bigcap \Phi$  then  $x = y$ . Let  $x, y \in A$ . Assume  $(0, x), (0, y) \in \bigcap \Phi$ .

Let us show that  $x, y = a$ . Assume  $x \neq a$  or  $y \neq a$ .

Case  $x \neq a$ .  $(0, x), (0, a)$  are contained in every element of  $\Phi$ . Then  $(0, x), (0, a) \in \bigcap \Phi$ . Take  $H = (\bigcap \Phi) \setminus \{(0, x)\}$ .

Let us show that  $H \in \Phi$ . (1)  $H \in \mathcal{P}(\mathbb{N} \times A)$ .

(2)  $(0, a) \in H$ .

(3) For all  $n \in \mathbb{N}$  and all  $b \in A$  if  $(n, b) \in H$  then  $(n+1, f(b)) \in H$ .

Proof. Let  $n \in \mathbb{N}$  and  $b \in A$ . Assume  $(n, b) \in H$ . Then  $(n+1, f(b)) \in \bigcap \Phi$ . We have  $(n+1, f(b)) \neq (0, x)$  (by FOUNDATIONS\_04\_8464577431863296). Indeed  $n+1 \neq 0$ . Hence  $(n+1, f(b)) \in H$ . Qed.

[prover vampire] Thus  $H \in \Phi$  (by a). [prover eprover] End.

Then  $(0, x)$  is not contained in every member of  $\Phi$ . Contradiction. End.

Case  $y \neq a$ .  $(0, y), (0, a)$  are contained in every element of  $\Phi$ . Then  $(0, y), (0, a) \in \bigcap \Phi$ . Take  $H = (\bigcap \Phi) \setminus \{(0, y)\}$ .

Let us show that  $H \in \Phi$ . (1)  $H \in \mathcal{P}(\mathbb{N} \times A)$ .

(2)  $(0, a) \in H$ . Indeed  $(0, a) \neq (0, y)$ .

(3) For all  $n \in \mathbb{N}$  and all  $b \in A$  if  $(n, b) \in H$  then  $(n+1, f(b)) \in H$ .

Proof. Let  $n \in \mathbb{N}$  and  $b \in A$ . Assume  $(n, b) \in H$ . Then  $(n+1, f(b)) \in \bigcap \Phi$ . We have  $(n+1, f(b)) \neq (0, y)$  (by FOUNDATIONS\_04\_8464577431863296). Indeed  $n+1 \neq 0$ . Hence  $(n+1, f(b)) \in H$ . Qed.

[prover vampire] Thus  $H \in \Phi$  (by a). [prover eprover] End.

Then  $(0, y)$  is not contained in every member of  $\Phi$ . Contradiction. End. End. Qed.

(2) For all  $n \in \Theta$  we have  $n+1 \in \Theta$ .

Proof. Let  $n \in \Theta$ . Then for all  $x, y \in A$  if  $(n, x), (n, y) \in \bigcap \Phi$  then  $x = y$ . Consider a  $b \in A$  such that  $(n, b) \in \bigcap \Phi$ . Then  $(n+1, f(b)) \in \bigcap \Phi$ .

Let us show that for all  $x, y \in A$  if  $(n+1, x), (n+1, y) \in \bigcap \Phi$  then  $x = f(b) = y$ . Let  $x, y \in A$ . Assume  $(n+1, x), (n+1, y) \in \bigcap \Phi$ . Suppose  $x \neq f(b)$  or  $y \neq f(b)$ .

Case  $x \neq f(b)$ . Take  $H = (\bigcap \Phi) \setminus \{(n+1, x)\}$ .

(i)  $H \in \mathcal{P}(\mathbb{N} \times A)$ .

(ii)  $(0, a) \in H$ . Indeed  $(0, a) \in \bigcap \Phi$  and  $(0, a) \notin \{(n+1, x)\}$ .

(iii) For all  $m \in \mathbb{N}$  and all  $z \in A$  if  $(m, z) \in H$  then  $(m+1, f(z)) \in H$ .

Proof. Let  $m \in \mathbb{N}$  and  $z \in A$ . Assume  $(m, z) \in H$ . Then  $(m, z) \in \bigcap \Phi$ . Hence  $(m+1, f(z)) \in \bigcap \Phi$ .  $(m+1, f(z)) \neq (n+1, x)$  (by FOUNDATIONS\_04\_8464577431863296). Indeed if  $(m+1, f(z)) = (n+1, x)$  then  $m = n$  and  $f(z) = x$  (by FOUNDATIONS\_04\_8464577431863296, ARITHMETIC\_01\_3604163883696128). Therefore  $(m+1, f(z)) \in H$ . Qed.

[prover vampire] Thus  $H \in \Phi$  (by a, i, ii, iii). [prover eprover] Therefore every element of  $\bigcap \Phi$  is contained in  $H$ . Consequently  $(n+1, x) \in H$ . Contradiction. End.

Case  $y \neq f(b)$ . Take  $H = (\bigcap \Phi) \setminus \{(n+1, y)\}$ .

(i)  $H \in \mathcal{P}(\mathbb{N} \times A)$ .

(ii)  $(0, a) \in H$ . Indeed  $(0, a) \in \bigcap \Phi$  and  $(0, a) \notin \{(n+1, y)\}$ .

(iii) For all  $m \in \mathbb{N}$  and all  $z \in A$  if  $(m, z) \in H$  then  $(m+1, f(z)) \in H$ .

Proof. Let  $m \in \mathbb{N}$  and  $z \in A$ . Assume  $(m, z) \in H$ . Then  $(m, z) \in \bigcap \Phi$ . Hence  $(m+1, f(z)) \in \bigcap \Phi$ .  $(m+1, f(z)) \neq (n+1, y)$  (by FOUNDATIONS\_04\_8464577431863296). Indeed if  $(m+1, f(z)) = (n+1, y)$  then  $m = n$  and  $f(z) = y$  (by FOUNDATIONS\_04\_8464577431863296, ARITHMETIC\_01\_3604163883696128). Therefore  $(m+1, f(z)) \in H$ . Qed.

[prover vampire] Thus  $H \in \Phi$  (by a, i, ii, iii). [prover eprover] Therefore every element of  $\bigcap \Phi$  is contained in  $H$ . Consequently  $(n+1, y) \in H$ . Contradiction. End.

Hence it is wrong that  $x \neq f(b)$  or  $y \neq f(b)$ . Consequently  $x = f(b) = y$ . End.

Therefore  $n + 1 \in \Theta$  (by a). Qed.

Consequently  $n \in \Theta$  for every  $n \in \mathbb{N}$  (by ARITHMETIC\_01\_4764664342773760). Qed.

Define  $\varphi(n) = \text{“choose } x \in A \text{ such that } (n, x) \in \bigcap \Phi \text{ in } x\text{”}$  for  $n \in \mathbb{N}$ .

(1) Then  $\varphi$  is a map from  $\mathbb{N}$  to  $A$  and we have  $\varphi(0) = a$ .

(2) For all  $n \in \mathbb{N}$  we have  $\varphi(n + 1) = f(\varphi(n))$ .

Proof. Let  $n \in \mathbb{N}$ . Take  $x \in A$  such that  $\varphi(n) = x$ . Then  $(n, x) \in \bigcap \Phi$ . Hence  $(n + 1, f(\varphi(n))) = (n + 1, f(x)) \in \bigcap \Phi$ . Thus  $\varphi(n + 1) = f(\varphi(n))$  (by a). Qed.  $\square$

**Theorem (Dedekind’s Recursion Theorem: Uniqueness).** Let  $A$  be a set and  $a \in A$  and  $f : A \rightarrow A$ . Let  $\varphi, \varphi' : \mathbb{N} \rightarrow A$ . Assume that  $\varphi$  and  $\varphi'$  are recursively defined by  $a$  and  $f$ . Then  $\varphi = \varphi'$ .

*Proof.* Define  $\Phi = \{n \in \mathbb{N} \mid \varphi(n) = \varphi'(n)\}$ .

(1)  $\Phi$  contains 0. Indeed  $\varphi(0) = a = \varphi'(0)$ .

(2) For all  $n \in \Phi$  we have  $n + 1 \in \Phi$ .

Proof. Let  $n \in \Phi$ . Then  $\varphi(n) = \varphi'(n)$ . Hence  $\varphi(n + 1) = f(\varphi(n)) = f(\varphi'(n)) = \varphi'(n + 1)$ . Qed.

Thus  $\Phi$  contains every natural number. Consequently  $\varphi(n) = \varphi'(n)$  for each  $n \in \mathbb{N}$ .  $\square$

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