

# Infinite Cardinal Numbers

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**Definition 0.1.** **Card** is the collection of all infinite cardinals.

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**Proposition 0.2.** Let  $\alpha$  be an infinite ordinal. Then  $|\text{succ}(\alpha)| = |\alpha|$ .

*Proof.* For any  $\beta \in \text{succ}(\alpha)$  we have  $\beta < \omega$  or  $\omega \leq \beta < \alpha$  or  $\beta = \alpha$ . Define

$$f(\beta) = \begin{cases} \text{succ}(\beta) & : \beta < \omega \\ \beta & : \omega \leq \beta < \alpha \\ 0 & : \beta = \alpha \end{cases}$$

for  $\beta \in \text{succ}(\alpha)$ .

Then  $f$  is a map from  $\text{succ}(\alpha)$  to  $\alpha$ . Indeed we can show that  $f(\beta) \in \alpha$  for all  $\beta \in \text{succ}(\alpha)$ .

*Proof.* Let  $\beta \in \text{succ}(\alpha)$ .

Case  $\beta < \omega$ . Then  $f(\beta) = \text{succ}(\beta) < \omega \leq \alpha$ . End.

Case  $\omega \leq \beta < \alpha$ . Then  $f(\beta) = \beta < \alpha$ . End.

Case  $\beta = \alpha$ . Then  $f(\beta) = 0 < \alpha$ . End. Qed.

$f$  is surjective onto  $\alpha$ . Indeed we can show that for any  $\beta \in \alpha$  there exists a  $\gamma \in \text{succ}(\alpha)$  such that  $\beta = f(\gamma)$ .

*Proof.* Let  $\beta \in \alpha$ . Then  $\beta = 0$  or  $0 < \beta < \omega$  or  $\beta \geq \omega$ .

Case  $\beta = 0$ . Then  $\beta = f(\alpha)$ . End.

Case  $0 < \beta < \omega$ . Take an ordinal  $\beta'$  such that  $\beta = \text{succ}(\beta')$ . Then  $\beta' < \omega$ . Hence  $\beta = f(\beta')$ . End.

Case  $\beta \geq \omega$ . Then  $\beta = f(\beta)$ . End. Qed.

$f$  is injective. Indeed we can show that for all  $\beta, \gamma \in \text{succ}(\alpha)$  if  $\beta \neq \gamma$  then  $f(\beta) \neq f(\gamma)$ .

*Proof.* Let  $\beta, \gamma \in \text{succ}(\alpha)$ . Assume  $\beta \neq \gamma$ .

Case  $\beta < \omega$ . If  $\gamma = \alpha$  then  $f(\beta) = \text{succ}(\beta) \neq 0 = f(\gamma)$ . If  $\omega \leq \gamma < \alpha$  then  $f(\beta) = \text{succ}(\beta) < \omega \leq \gamma = f(\gamma)$ . End.

Case  $\omega \leq \beta < \alpha$ . If  $\gamma = \alpha$  then  $f(\beta) = \beta \geq \omega > 0 = f(\gamma)$ . If  $\gamma < \omega$  then  $f(\beta) = \beta \geq \omega > \text{succ}(\gamma) = f(\gamma)$ . End.

Case  $\beta = \alpha$ . If  $\gamma < \omega$  then  $f(\beta) = 0 \neq \text{succ}(\gamma) = f(\gamma)$ . If  $\omega \leq \gamma < \alpha$  then  $f(\beta) = 0 < \omega \leq \gamma = f(\gamma)$ . End. Qed.

Hence  $f$  is a bijection between  $\text{succ}(\alpha)$  and  $\alpha$ . Therefore  $\text{succ}(\alpha)$  and  $\alpha$  are equinumerous. Consequently  $|\text{succ}(\alpha)| = |\alpha|$ .  $\square$

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**Proposition 0.3.** Every infinite cardinal is a limit ordinal.

*Proof.* Let  $\kappa$  be an infinite cardinal. Suppose that  $\kappa$  is not a limit ordinal.  $\kappa \neq 0$ . Hence  $\kappa$  is a successor ordinal. Indeed  $\kappa$  is an ordinal. Thus we can take an ordinal  $\alpha$  such that  $\kappa = \text{succ}(\alpha)$ . We have  $\alpha > \kappa \geq \omega$ . Hence  $|\text{succ}(\alpha)| = |\alpha|$ . Thus  $\alpha < |\kappa|$  and  $\kappa$  is equinumerous to  $\kappa$ . Contradiction.  $\square$