

Little Gauß' Theorem

Naproche formalization:

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2024

This is a formalization of “Little Gauß' Theorem”, i.e. of the assertion that

$$\sum_{k=0}^n k = \frac{k(k+1)}{2}$$

for all $n \in \mathbb{N}$. In Naproche we can define the sum $\sum_{n=0}^k n$ by the function $\mathcal{G} : \mathbb{N} \rightarrow \mathbb{N}$ with $\mathcal{G}(0) = 0$ and $\mathcal{G}(n+1) = \mathcal{G}(n) + (n+1)$:

Signature. Let n be a natural number. $\mathcal{G}(n)$ is a natural number.

Axiom. $\mathcal{G}(0) = 0$.

Axiom. Let n be a natural number. Then $\mathcal{G}(n+1) = \mathcal{G}(n) + (n+1)$.

Then the theorem can be stated as follows.

Theorem (Little Gauß). For all natural numbers n we have

$$2 \cdot \mathcal{G}(n) = n \cdot (n+1).$$

Proof. Define $\Phi = \{n \in \mathbb{N} \mid 2 \cdot \mathcal{G}(n) = n \cdot (n+1)\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $n+1 \in \Phi$.

Proof. Let $n \in \Phi$. Then $2 \cdot \mathcal{G}(n) = n \cdot (n+1)$. Hence $2 \cdot \mathcal{G}(n+1) = 2 \cdot (\mathcal{G}(n) + (n+1)) = (2 \cdot \mathcal{G}(n)) + (2 \cdot (n+1)) = (n \cdot (n+1)) + (2 \cdot (n+1)) = ((n+1) \cdot n) + ((n+1) \cdot 2) = (n+1) \cdot (n+2) = (n+1) \cdot ((n+1) + 1)$. Therefore $n+1 \in \Phi$. Qed.

Thus Φ contains every natural number. Consequently $2 \cdot \mathcal{G}(n) = n \cdot (n+1)$ for every $n \in \mathbb{N}$. \square

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