Furstenberg's proof of the infinitude of primes

ℕaproche formalization:

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This is a formalization of Furstenberg's topological proof of the infinitude of primes [**Furstenberg1955**]. On mid-range hardware ℕaproche needs approximately 5 Minutes to verify this formalization plus approximately 40 minutes to verify the library files it depends on.

[readtex `libraries/source/arithmetics/primes.ftl.tex`]

[readtex `libraries/source/arithmetics/nat-is-a-set.ftl.tex`]

[readtex `libraries/source/set-theory/zf.ftl.tex`]

[readtex `libraries/source/foundations/closure-under-finite-unions.ftl.tex`]

The central idea of Furstenberg's proof is to define a certain topology on $\mathbb{N}$ from the properties of which we can deduce that the set of primes is infinite.[1]

Let n, m, k denote natural numbers. Let p, q denote nonzero natural numbers.

Definition 1. Let A be a subset of $\mathbb{N}$. $A^{\complement} = \mathbb{N} \setminus A$.

Let the complement of A stand for $A^{\complement}$.

Lemma 2. The complement of any subset of $\mathbb{N}$ is a subset of $\mathbb{N}$.

Towards a suitable topology on $\mathbb{N}$ let us define *arithmetic sequences* $\mathrm{N}_{n,q}$ on $\mathbb{N}$.

Definition 3. $\mathrm{N}_{n,q} = \{m \in \mathbb{N} \mid m \equiv n \pmod q\}$.

This allows us to define the *evenly spaced natural number topology* on $\mathbb{N}$, whose open sets are defined as follows.

[1] Actually, Furstenberg's proof makes use of a topology on $\mathbb{Z}$. But this topology can as well be restricted to $\mathbb{N}$ without substantially changing the proof.

Definition 4. Let U be a subset of $\mathbb{N}$. U is open iff for any $n \in U$ there exists a q such that $\mathrm{N}_{n,q} \subseteq U$.

Definition 5. A system of open sets is a system of sets S such that every element of S is an open subset of $\mathbb{N}$.

Lemma 6. Every system of open sets is a set.

Proof. Let S be a system of open sets. Then $S \subseteq \mathcal{P}(\mathbb{N})$. Hence S is a set. $\square$

We can show that the open sets indeed form a topology on $\mathbb{N}$.

Lemma 7. $\mathbb{N}$ and $\emptyset$ are open.

Lemma 8. Let U, V be open subsets of $\mathbb{N}$. Then $U \cap V$ is open.

Proof. Let $n \in U \cap V$. Take a q such that $\mathrm{N}_{n,q} \subseteq U$. Take a p such that $\mathrm{N}_{n,p} \subseteq V$. Then $p \cdot q \neq 0$.

Let us show that $\mathrm{N}_{n,p \cdot q} \subseteq U \cap V$. Let $m \in \mathrm{N}_{n,p \cdot q}$. We have $m \equiv n \pmod{p \cdot q}$. Hence $m \equiv n \pmod{p}$ and $m \equiv n \pmod{q}$. Thus $m \in \mathrm{N}_{n,p}$ and $m \in \mathrm{N}_{n,q}$. Therefore $m \in U$ and $m \in V$. Consequently $m \in U \cap V$. End. $\square$

Lemma 9. Let S be a system of open sets. Then $\bigcup S$ is open.

Proof. Let $n \in \bigcup S$. Take a set M such that $n \in M \in S$. Consider a q such that $\mathrm{N}_{n,q} \subseteq M$. Then $\mathrm{N}_{n,q} \subseteq \bigcup S$. $\square$

Now that we have a topology of open sets on $\mathbb{N}$, we can continue with a characterization of closed sets whose key property is that they are closed under finite unions.

Definition 10. Let A be a subset of $\mathbb{N}$. A is closed iff $A^{\complement}$ is open.

Definition 11. A system of closed sets is a system of sets S such that every element of S is a closed subset of $\mathbb{N}$.

Lemma 12. Every system of closed sets is a set.

Proof. Let S be a system of closed sets. Then $S \subseteq \mathcal{P}(\mathbb{N})$. $\mathcal{P}(\mathbb{N})$ is a set. Hence S is a set. $\square$

Lemma 13. Let S be a finite system of closed sets. Then $\bigcup S$ is closed.

Proof. Define $C = \{X \mid X \text{ is a closed subset of } \mathbb{N}\}$.

Let us show that $A \cup B \in C$ for any $A, B \in C$. Let $A, B \in C$. Then A, B are closed subsets of $\mathbb{N}$. We have $((A \cup B)^{\complement}) = A^{\complement} \cap B^{\complement}$. $A^{\complement}$ and $B^{\complement}$ are open. Hence $A^{\complement} \cap B^{\complement}$ is open. Thus $A \cup B$ is a closed subset of $\mathbb{N}$. End.

Therefore C is closed under finite unions. Consequently $\bigcup S \in C$. Indeed S is a subset of C. $\square$

An important step towards Furstenberg's proof is to show that arithmetic sequences are closed.

Lemma 14. $\mathrm{N}_{n,q}$ is closed.
Proof. Let $m \in (\mathrm{N}_{n,q})^{\complement}$.

Let us show that $\mathrm{N}_{m,q} \subseteq (\mathrm{N}_{n,q})^{\complement}$. Let $k \in \mathrm{N}_{m,q}$. Assume $k \notin (\mathrm{N}_{n,q})^{\complement}$. Then $k \equiv m \pmod{q}$ and $n \equiv k \pmod{q}$. Hence $m \equiv n \pmod{q}$. Therefore $m \in \mathrm{N}_{n,q}$. Contradiction. End. □

Identifying each prime number p with the arithmetic sequence $\mathrm{N}_{0,p}$ yields a bijection between the set $\mathbb{P}$ of all prime numbers and the set P of all such sequences $\mathrm{N}_{0,p}$. Thus to show that there are infinitely many primes it suffices to show that P is infinite.

Definition 15. $\mathrm{P} = \{\mathrm{N}_{0,p} \mid p \in \mathbb{P}\}$.

Lemma 16. P is a system of closed sets.

Proof. $\mathrm{N}_{0,p}$ is a closed subset of $\mathbb{N}$ for every $p \in \mathbb{P}$. □

Lemma 17. P is a set that is equinumerous to $\mathbb{P}$.

Proof. (1) P is a set. Indeed $\mathrm{P} \subseteq \mathcal{P}(\mathbb{N})$.

(2) P is equinumerous to $\mathbb{P}$.
Proof. Define $f(p) = \mathrm{N}_{0,p}$ for $p \in \mathbb{P}$.

Let us show that f is injective. Let $p, q \in \mathbb{P}$. Assume $f(p) = f(q)$. Then $\mathrm{N}_{0,p} = \mathrm{N}_{0,q}$. We have $\mathrm{N}_{0,p} = \{m \in \mathbb{N} \mid m \equiv 0 \pmod{p}\}$ and $\mathrm{N}_{0,q} = \{m \in \mathbb{N} \mid m \equiv 0 \pmod{q}\}$.

We can show that for all $m \in \mathbb{N}$ we have $p \mid m$ iff $q \mid m$. Let $m \in \mathbb{N}$. Then $m \equiv 0 \pmod{p}$ iff $m \equiv 0 \pmod{q}$. Thus $m \bmod p = 0 \bmod p$ iff $m \bmod q = 0 \bmod q$. We have $0 \bmod p = 0 = 0 \bmod q$. Hence $m \bmod p = 0$ iff $m \bmod q = 0$. Therefore $p \mid m$ iff $q \mid m$. End.

Consequently $p = q$. End.

f is surjective onto P. Thus f is a bijection between $\mathbb{P}$ and P. Qed. □

Theorem 18 (Furstenberg). $\mathbb{P}$ is infinite.

Proof. $\bigcup \mathrm{P}$ is a subset of $\mathbb{N}$.

Let us show that for any $n \in \mathbb{N}$ we have $n \in \bigcup \mathrm{P}$ iff n has a prime divisor. Let $n \in \mathbb{N}$.

If n has a prime divisor then n belongs to $\bigcup \mathrm{P}$.
Proof. Assume n has a prime divisor. Take a prime divisor p of n. We have $\mathrm{N}_{0,p} \in \mathrm{P}$. Hence $n \in \mathrm{N}_{0,p}$. Qed.

If n belongs to $\bigcup \mathrm{P}$ then n has a prime divisor.
Proof. Assume that n belongs to $\bigcup \mathrm{P}$. Take a prime number r such that

$n \in \mathrm{N}_{0,r}$. Hence $n \equiv 0 \pmod{r}$. Thus $n \bmod r = 0 \bmod r = 0$. Therefore r is a prime divisor of n. Qed. End.

Hence For all $n \in \mathbb{N}$ we have $n \in (\bigcup \mathrm{P})^{\complement}$ iff n has no prime divisor. 1 has no prime divisor and any natural number having no prime divisor is equal to 1. Therefore $(\bigcup \mathrm{P})^{\complement} = \{1\}$. Indeed $((\bigcup \mathrm{P})^{\complement}) \subseteq \{1\}$ and $\{1\} \subseteq (\bigcup \mathrm{P})^{\complement}$.

P is infinite.
Proof by contradiction. Assume that P is finite. Then $\bigcup \mathrm{P}$ is closed and $(\bigcup \mathrm{P})^{\complement}$ is open. Take a p such that $\mathrm{N}_{1,p} \subseteq (\bigcup \mathrm{P})^{\complement}$. $1 + p$ is an element of $\mathrm{N}_{1,p}$. Indeed $1 + p \equiv 1 \pmod{p}$ (by `ARITHMETIC_08_5984712287846400`). $1 + p$ is not equal to 1. Hence $1 + p \notin (\bigcup \mathrm{P})^{\complement}$. Contradiction. Qed. $\square$