Maximum modulus principle

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1 Introduction

We formalize that the maximum modulus principle in complex analysis is implied by the open mapping theorem and the identity theorem. Note that we do not give standard definitions of complex analytic notions, but we only postulate axioms used in the proof. The axioms are satisfied over the standard complex numbers but they also have non-standard models. The formalization checks in under one minute on a modest laptop.

We use general set theoretic preliminaries:

[readtex preliminaries.ftl.tex]

2 Real and complex numbers

Signature 1. A complex number is a mathematical object.

Let z, w denote complex numbers.

Definition 2. $\mathbb{C}$ is the collection of all complex numbers.

Signature 3. A real number is a complex number.

Let x, y denote real numbers.

Signature 4. $|z|$ is a real number.

Signature 5. x is positive is an atom.

Let ε, δ denote positive real numbers.

Signature 6. $x < y$ is an atom.

Let $x > y$ stand for $y < x$. Let $x \leq y$ stand for $x = y$ or $x < y$.

Axiom 7. If $x < y$ then not $y < x$.

3 Open balls

Let z, w denote complex numbers. Let ε, δ denote positive real numbers.

Signature 8. Let ε be a positive real number. $B_\varepsilon(z)$ is a subset of $\mathbb{C}$ that contains z.

Axiom 9. $|z| < |w|$ for some element w of $B_\varepsilon(z)$.

Let M denote a subset of $\mathbb{C}$.

Definition 10. M is open iff for every element z of M there exists ε such that $B_\varepsilon(z)$ is a subset of M.

Axiom 11. $B_\varepsilon(z)$ is open.

Signature 12. A region is an open subset of $\mathbb{C}$.

Signature 13. Let M be a region. M is simply connected is an atom.

4 Holomorphic functions

Let z, w denote complex numbers. Let ε, δ denote positive real numbers.

Signature 14. A holomorphic function is a function f such that $\mathrm{dom}(f) \subseteq \mathbb{C}$ and $f[\mathrm{dom}(f)] \subseteq \mathbb{C}$.

Let f denote a holomorphic function.

Definition 15. A local maximal point of f is an element z of the domain of f such that there exists ε such that $B_\varepsilon(z)$ is a subset of the domain of f and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Definition 16. Let U be a subset of the domain of f. f is constant on U iff there exists z such that $f(w) = z$ for every element w of U.

Let f is constant stand for f is constant on the domain of f.

Axiom 17 (Open Mapping Theorem). Assume f is a holomorphic function and $B_\varepsilon(z)$ is a subset of the domain of f. If f is not constant on $B_\varepsilon(z)$ then $f[B_\varepsilon(z)]$ is open.

Axiom 18 (Identity theorem). Assume f is a holomorphic function and the domain of f is a region. Assume that $B_\varepsilon(z)$ is a subset of the domain of f. If f is constant on $B_\varepsilon(z)$ then f is constant.

Proposition 19 (Maximum modulus principle). Assume f is a holomorphic function and the domain of f is a region. If f has a local maximal point then f is constant.

Proof. Let z be a local maximal point of f. Take ε such that $B_\varepsilon(z)$ is a subset of $\mathrm{dom}(f)$ and $|f(w)| \leq |f(z)|$ for every element w of $B_\varepsilon(z)$.

Let us show that f is constant on $B_\varepsilon(z)$. Proof by contradiction. Assume the contrary. Then $f[B_\varepsilon(z)]$ is open. We can take δ such that $B_\delta(f(z))$

is a subset of $f[B_\varepsilon(z)]$. Therefore there exists an element w of $B_\varepsilon(z)$ such that $|f(z)| < |f(w)|$. Contradiction. End.

Hence f is constant. $\square$