

## Exponentiation and Ordering

ARITHMETIC\_09\_3373702288769024

**Proposition 0.1.** Let  $n, m, k$  be natural numbers. Assume  $k \neq 0$ . Then  $n < m$  iff  $n^k < m^k$ .

*Proof.* Case  $n < m$ . Define  $\Phi = \{k' \in \mathbb{N} \mid \text{if } k' > 1 \text{ then } n^{k'} < m^{k'}\}$ .

(1)  $\Phi$  contains 0.

(2)  $\Phi$  contains 1.

(3)  $\Phi$  contains 2.

*Proof.* Case  $n = 0$  or  $m = 0$ . Obvious.

Case  $n, m \neq 0$ . Then  $n \cdot n < n \cdot m < m \cdot m$ . Hence  $n^2 = n \cdot n < n \cdot m < m \cdot m = m^2$ . End. Qed.

(4) For all  $k' \in \Phi$  we have  $k' + 1 \in \Phi$ .

*Proof.* Let  $k' \in \Phi$ .

Let us show that if  $k' + 1 > 1$  then  $n^{k'+1} < m^{k'+1}$ . Assume  $k' + 1 > 1$ . Then  $n^{k'} < m^{k'}$ . Indeed  $k' \neq 0$  and if  $k' = 1$  then  $n^{k'} < m^{k'}$ .

Case  $k' \leq 1$ . Then  $k' = 0$  or  $k' = 1$ . Hence  $k' + 1 = 1$  or  $k' + 1 = 2$ . Thus  $k' + 1 \in \Phi$ . Therefore  $n^{k'+1} < m^{k'+1}$ . End.

Case  $k' > 1$ . Case  $n = 0$ . Then  $m \neq 0$ . Hence  $n^{k'+1} = 0 < m^{k'} \cdot m = m^{k'+1}$ . Thus  $n^{k'+1} < m^{k'+1}$ . End.

Case  $n \neq 0$ . Then  $n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m$ . Indeed  $n^{k'} < m^{k'} \neq 0$ . Take  $A = n^{k'+1}$  and  $B = m^{k'+1}$ . Then  $A = n^{k'+1} = n^{k'} \cdot n < m^{k'} \cdot n < m^{k'} \cdot m = m^{k'+1} = B$ . Take  $X = m^{k'} \cdot n$  and  $Y = m^{k'} \cdot m$ . Then  $A < X < Y = B$ . Hence  $A < B$ . Thus  $n^{k'+1} < m^{k'+1}$ . End. End.

Hence  $n^{k'+1} < m^{k'+1}$ . Indeed  $k' \leq 1$  or  $k' > 1$ . End.

Thus  $k' + 1 \in \Phi$ . Qed.

Therefore every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_47 64664342773760). Consequently  $n^k < m^k$ . End.

Case  $n^k < m^k$ . Define  $\Psi = \{k' \in \mathbb{N} \mid \text{if } n \geq m \text{ then } n^{k'} \geq m^{k'}\}$ .

(1)  $\Psi$  contains 0.

(2) For all  $k' \in \Psi$  we have  $k' + 1 \in \Psi$ .

*Proof.* Let  $k' \in \Psi$ .

Let us show that if  $n \geq m$  then  $n^{k'+1} \geq m^{k'+1}$ . Assume  $n \geq m$ . Then  $n^{k'} \geq m^{k'}$ . Hence  $n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m$ . Take  $A = n^{k'+1}$  and  $B = m^{k'+1}$ . Thus  $A = n^{k'+1} = n^{k'} \cdot n \geq m^{k'} \cdot n \geq m^{k'} \cdot m = m^{k'+1} = B$ . Therefore  $n^{k'+1} = A \geq B = m^{k'+1}$ . End.

Hence  $k' + 1 \in \Psi$ . Qed.

Thus every natural number is contained in  $\Psi$  (by ARITHMETIC\_01\_476466 4342773760). Therefore if  $n \geq m$  then  $n^k \geq m^k$ . Consequently  $n < m$ . End.  $\square$

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**Corollary 0.2.** Let  $n, m, k$  be natural numbers. Assume  $k \neq 0$ . Then

$$n^k = m^k \quad \text{implies} \quad n = m.$$

*Proof.* Assume  $n^k = m^k$ . Suppose  $n \neq m$ . Then  $n < m$  or  $m < n$ . If  $n < m$  then  $n^k < m^k$ . If  $m < n$  then  $m^k < n^k$ . Thus  $n^k \neq m^k$ . Contradiction.  $\square$

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**Corollary 0.3.** Let  $n, m, k$  be natural numbers. Assume  $k \neq 0$ . Then

$$n^k \leq m^k \quad \text{iff} \quad n \leq m.$$

*Proof.* If  $n^k < m^k$  then  $n < m$ . If  $n^k = m^k$  then  $n = m$ .

If  $n < m$  then  $n^k < m^k$ . If  $n = m$  then  $n^k = m^k$ .  $\square$

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**Proposition 0.4.** Let  $n, m, k$  be natural numbers. Assume  $k > 1$ . Then

$$n < m \quad \text{iff} \quad k^n < k^m.$$

*Proof.* Case  $n < m$ . Define  $\Phi = \{m' \in \mathbb{N} \mid \text{if } n < m' \text{ then } k^n < k^{m'}\}$ .

(1)  $\Phi$  contains 0.

(2) For all  $m' \in \Phi$  we have  $m' + 1 \in \Phi$ .

*Proof.* Let  $m' \in \Phi$ .

Let us show that if  $n < m' + 1$  then  $k^n < k^{m'+1}$ . Assume  $n < m' + 1$ . Then  $n \leq m'$ . We have  $k^{m'} \cdot 1 < k^{m'} \cdot k$ . Indeed  $k^{m'} \neq 0$ .

Case  $n = m'$ . Take  $A = k^n$  and  $B = k^{m'+1}$ . Then  $A = k^n = k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$ . Hence  $k^n = A < B = k^{m'+1}$ . End.

Case  $n < m'$ . Take  $A = k^n$  and  $B = k^{m'+1}$ . Then  $A = k^n < k^{m'} < k^{m'} \cdot k = k^{m'+1} = B$ . Hence  $k^n = A < B = k^{m'+1}$ . End. Qed. Qed.

Hence every natural number is contained in  $\Phi$  (by ARITHMETIC\_01\_4764 664342773760). Thus  $k^n < k^m$ . End.

Case  $k^n < k^m$ . Define  $\Psi = \{n' \in \mathbb{N} \mid \text{if } n' \geq m \text{ then } k^{n'} \geq k^m\}$ .

(1) 0 is contained in  $\Psi$ .

(2) For all  $n' \in \Psi$  we have  $n' + 1 \in \Psi$ .

Proof. Let  $n' \in \Psi$ .

Let us show that if  $n' + 1 \geq m$  then  $k^{n'+1} \geq k^m$ . Assume  $n' + 1 \geq m$ .

Case  $n' + 1 = m$ . Obvious.

Case  $n' + 1 > m$ . Then  $n' \geq m$ . Hence  $k^{n'} \geq k^m$ . We have  $k^{n'} \cdot 1 \leq k^{n'} \cdot k$ . Indeed  $1 \leq k$  and  $k^{n'} \neq 0$ . Take  $A = k^m$  and  $B = k^{n'+1}$ . Then  $A = k^m \leq k^{n'} = k^{n'} \cdot 1 \leq k^{n'} \cdot k = k^{n'+1} = B$ . Hence  $k^m = A \leq B = k^{n'+1}$ . End. Qed. Qed.

Thus every natural number is contained in  $\Psi$  (by ARITHMETIC\_01\_476466 4342773760). Therefore if  $n \geq m$  then  $k^n \geq k^m$ . Consequently  $n < m$ . End.  $\square$

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**Corollary 0.5.** Let  $n, m, k$  be natural numbers. Assume  $k > 1$ . If  $k^n = k^m$  then  $n = m$ .

*Proof.* Assume  $k^n = k^m$ . Suppose  $n \neq m$ . Then  $n < m$  or  $m < n$ . If  $n < m$  then  $k^n < k^m$ . If  $m < n$  then  $k^m < k^n$ . Thus  $k^n \neq k^m$ . Contradiction.  $\square$

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**Corollary 0.6.** Let  $n, m, k$  be natural numbers. Assume  $k > 1$ . Then  $n \leq m$  iff  $k^n \leq k^m$ .

*Proof.* If  $n \leq m$  then  $k^n \leq k^m$ .

If  $k^n = k^m$  then  $n = m$ . If  $k^n < k^m$  then  $n < m$ .  $\square$

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**Proposition 0.7.** Let  $n$  be a natural number. Then  $(n+1)^2 = (n^2 + (2 \cdot n)) + 1$ .

*Proof.* We have

$$\begin{aligned} (n+1)^2 &= (n+1) \cdot (n+1) \\ &= ((n+1) \cdot n) + (n+1) \\ &= ((n \cdot n) + n) + (n+1) \end{aligned}$$

$$\begin{aligned}
&= (n^2 + n) + (n + 1) \\
&= ((n^2 + n) + n) + 1 \\
&= (n^2 + (n + n)) + 1 \\
&= (n^2 + (2 \cdot n)) + 1.
\end{aligned}$$

□

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**Proposition 0.8.** Let  $n$  be a natural number. Assume  $n \geq 3$ . Then  $n^2 > (2 \cdot n) + 1$ .

*Proof.* (a) Define  $\Phi = \{n' \in \mathbb{N}_{\geq 3} \mid n'^2 > (2 \cdot n') + 1\}$ .

(1)  $\Phi$  contains 3. Indeed  $3^2 > (2 \cdot 3) + 1$ .

(2) For all  $n' \in \Phi$  we have  $n' + 1 \in \Phi$ .

*Proof.* Let  $n' \in \Phi$ . Then  $n' \geq 3$ .

Take  $V = (n'^2 + (2 \cdot n')) + 1$  and  $W = (((2 \cdot n') + 1) + (2 \cdot n')) + 1$ . Then  $V > W$ .

*Proof.* We have  $n'^2 > (2 \cdot n') + 1$ . Hence  $n'^2 + (2 \cdot n') > ((2 \cdot n') + 1) + (2 \cdot n')$  (by ARITHMETIC\_04\_7354062662008832). Qed.

Take  $X = ((2 \cdot n') + (2 \cdot n')) + 1$ . Then  $W > X$ .

*Proof.* We have  $((2 \cdot n') + 1) + (2 \cdot n') > (2 \cdot n') + (2 \cdot n')$ . Indeed  $(2 \cdot n') + 1 > 2 \cdot n'$ . Qed.

Take  $Y = (2 \cdot (n' + n')) + 1$  and  $Z = (2 \cdot (n' + 1)) + 1$ . Then  $Y > Z$ .

*Proof.* We have  $n' + n' > n' + 1$  and  $2 \neq 0$ . Thus  $2 \cdot (n' + n') > 2 \cdot (n' + 1)$  (by ARITHMETIC\_06\_5048640368279552). Indeed  $n' + n'$  and  $n' + 1$  are natural numbers. Qed.

Then  $(n' + 1)^2 = V > W > X = Y > Z = (2 \cdot (n' + 1)) + 1$ . Hence  $(n' + 1)^2 > (2 \cdot (n' + 1)) + 1$ . Qed.

Therefore  $\Phi$  contains every element of  $\mathbb{N}_{\geq 3}$  (by ARITHMETIC\_04\_49765992 69113856, 1, 2). Consequently  $n^2 > (2 \cdot n) + 1$  (by a). □

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**Proposition 0.9.** Let  $n$  be a natural number. Assume  $n \geq 5$ . Then  $2^n > n^2$ .

*Proof.* (a) Define  $\Phi = \{n' \in \mathbb{N}_{\geq 5} \mid 2^{n'} > n'^2\}$ .

(1)  $\Phi$  contains 5. Indeed we can show that  $2^5 > 5^2$ . We have  $2^5 = 2 \cdot (2 \cdot (2 \cdot 2)) = 8 \cdot 4 = (5+3) \cdot 4 = (5 \cdot 4) + (3 \cdot 4) = (5 \cdot 4) + ((5 \cdot 2) + 2) = (5 \cdot (4+2)) + 2 =$

$(5 \cdot 6) + 2 > 5 \cdot 6 > 5 \cdot 5 = 5^2$ . Indeed  $((5 \cdot 4) + ((5 \cdot 2) + 2)) = (5 \cdot (4 + 2)) + 2$ .  
End.

(2) For all  $n' \in \Phi$  we have  $n' + 1 \in \Phi$ .

Proof. Let  $n' \in \Phi$ . Then  $n' \geq 5$  and  $2^{n'} > n'^2$ .

Take  $V = 2^{n'} \cdot 2$  and  $W = n'^2 \cdot 2$ . Then  $V > W$ . Indeed  $2 \neq 0$ .

Take  $X = n'^2 + n'^2$ . Then  $W = X$ .

Take  $Y = n'^2 + ((2 \cdot n') + 1)$ . Then  $X > Y$ . Indeed  $n'^2 > (2 \cdot n') + 1$ .

Take  $Z = (n' + 1)^2$ . Then  $Y = Z$ .

Then  $2^{n'+1} = V > W = X > Y = Z = (n' + 1)^2$ . Hence  $2^{n'+1} > (n' + 1)^2$ .  
Qed.

Therefore  $\Phi$  contains every element of  $\mathbb{N}_{\geq 5}$  (by ARITHMETIC\_04\_49765992 69113856). Consequently  $2^n > n^2$  (by a).  $\square$