

# Classes

## Sub- and Superclasses

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**Definition 0.1.** Let  $A$  be a class. A subclass of  $A$  is a class  $B$  such that every element of  $B$  is an element of  $A$ .

Let  $B \subseteq A$  stand for  $B$  is a subclass of  $A$ . Let  $B \subset A$  stand for  $B \subseteq A$ .

Let a superclass of  $B$  stand for a class  $A$  such that  $B \subseteq A$ . Let  $B \supseteq A$  stand for  $B$  is a superclass of  $A$ . Let  $B \supset A$  stand for  $B \subseteq A$ .

Let a proper subclass of  $A$  stand for a subclass  $B$  of  $A$  such that  $B \neq A$ . Let  $B \subsetneq A$  stand for  $B$  is a proper subclass of  $A$ .

Let a proper superclass of  $B$  stand for a superclass  $A$  of  $B$  such that  $A \neq B$ . Let  $B \subsetneq A$  stand for  $B$  is a proper superclass of  $A$ .

Let  $A$  includes  $B$  stand for  $B \subseteq A$ . Let  $B$  is included in  $A$  stand for  $B \subseteq A$ .

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**Proposition 0.2.** Let  $A$  be a class. Then  $A \subseteq A$ .

*Proof.* Every element of  $A$  is contained in  $A$ . Therefore  $A \subseteq A$ . □

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**Proposition 0.3.** Let  $A, B, C$  be classes. If  $A \subseteq B$  and  $B \subseteq C$  then  $A \subseteq C$ .

*Proof.* Assume  $A \subseteq B$  and  $B \subseteq C$ . Then every element of  $A$  is contained in  $B$  and every element of  $B$  is contained in  $C$ . Hence every element of  $A$  is contained in  $C$ . Thus  $A \subseteq C$ . □

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**Proposition 0.4.** Let  $A, B$  be classes. If  $A \subseteq B$  and  $B \subseteq A$  then  $A = B$ .

*Proof.* Assume  $A \subseteq B$  and  $B \subseteq A$ . Then every element of  $A$  is contained in  $B$  and every element of  $B$  is contained in  $A$ . Hence  $A = B$ . □

## The Empty Class

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**Definition 0.5.** Let  $A$  be a class.  $A$  is empty iff  $A$  has no elements.

Let  $A$  is nonempty stand for  $A$  is not empty.

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**Definition 0.6.**  $\emptyset = \{x \mid x \neq x\}$ .

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**Proposition 0.7.** Let  $A$  be a class.  $A$  is empty iff  $A = \emptyset$ .

*Proof.* We can show that  $\emptyset$  is empty. Indeed any element  $x$  of  $\emptyset$  is nonequal to  $x$ . Hence if  $A = \emptyset$  then  $A$  is empty. If  $A$  is empty then  $A$  and  $\emptyset$  have no elements. Hence if  $A$  is empty then  $A \subseteq \emptyset$  and  $\emptyset \subseteq A$ . Thus if  $A$  is empty then  $A = \emptyset$ .  $\square$

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**Corollary 0.8.**  $\emptyset$  is empty.

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**Corollary 0.9.** Let  $A$  be a class. Then  $\emptyset \subseteq A$ .

*Proof.*  $\emptyset$  has no elements. Hence every element of  $\emptyset$  is contained in  $A$ .  $\square$

## Unordered Pairs

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**Definition 0.10.** Let  $a, b$  be objects. The unordered pair of  $a$  and  $b$  is  $\{x \mid x = a \text{ or } x = b\}$ .

Let  $\{a, b\}$  stand for the unordered pair of  $a$  and  $b$ .

**Definition 0.11.** An unordered pair is a class  $A$  such that  $A = \{a, b\}$  for some distinct objects  $a, b$ .

**Definition 0.12.** Let  $a$  be an object. The singleton class of  $a$  is  $\{x \mid x = a\}$ .

Let  $\{a\}$  stand for the singleton class of  $a$ .

**Definition 0.13.** A singleton class is a class  $A$  such that  $A = \{a\}$  for some object  $a$ .

**Proposition 0.14.** Let  $a, a', b, b'$  be objects. Assume  $\{a, b\} = \{a', b'\}$ . Then  $(a = a' \text{ and } b = b')$  or  $(a = b' \text{ and } b = a')$ .

*Proof.* We have  $a = a'$  or  $a = b'$ . If  $a = a'$  then  $b = b'$ . If  $a = b'$  then  $b = a'$ . Hence  $(a = a' \text{ and } b = b')$  or  $(a = b' \text{ and } b = a')$ .  $\square$

**Corollary 0.15.** Let  $a, a'$  be objects. If  $\{a\} = \{a'\}$  then  $a = a'$ .

**Definition 0.16.** Let  $A$  be a class. A unique element of  $A$  is an element  $a$  of  $A$  such that for each  $x \in A$  we have  $x = a$ .

**Proposition 0.17.** Let  $A$  be a class. Then  $A$  has a unique element iff  $A = \{a\}$  for some object  $a$ .

## Unions, Intersections, Complements

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**Definition 0.18.** Let  $A, B$  be classes. The union of  $A$  and  $B$  is  $\{x \mid x \in A \text{ or } x \in B\}$ .

Let  $A \cup B$  stand for the union of  $A$  and  $B$ .

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**Definition 0.19.** Let  $A, B$  be classes. The intersection of  $A$  and  $B$  is  $\{x \mid x \in A \text{ and } x \in B\}$ .

Let  $A \cap B$  stand for the intersection of  $A$  and  $B$ .

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**Definition 0.20.** Let  $A, B$  be classes. The complement of  $B$  in  $A$  is  $\{x \mid x \in A \text{ and } x \notin B\}$ .

Let  $A \setminus B$  stand for the complement of  $B$  in  $A$ .

## Disjoint Classes

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**Definition 0.21.** Let  $A, B$  be classes.  $A$  and  $B$  are disjoint iff  $A$  and  $B$  have no common elements.

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**Proposition 0.22.** Let  $A, B$  be classes. Then  $A$  and  $B$  are disjoint iff  $A \cap B$  is empty.