

## Divisibility

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**Definition 0.1.** Let  $n, m$  be natural numbers.  $n$  divides  $m$  iff there exists a natural number  $k$  such that  $n \cdot k = m$ .

Let  $m$  is divisible by  $n$  stand for  $n$  divides  $m$ . Let  $n \mid m$  stand for  $n$  divides  $m$ . Let  $n \nmid m$  stand for  $n$  does not divide  $m$ .

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**Lemma 0.2.** Let  $n, m$  be natural numbers.  $n$  divides  $m$  iff there exists a natural number  $k$  such that  $k \cdot n = m$ .

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**Definition 0.3.** Let  $n$  be a natural number. A factor of  $n$  is a natural number that divides  $n$ .

Let a divisor of  $n$  stand for a factor of  $n$ .

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**Definition 0.4.** Let  $n$  be a natural number. A trivial divisor of  $n$  is a divisor  $m$  of  $n$  such that  $m = 1$  or  $m = n$ .

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**Definition 0.5.** Let  $n$  be a natural number. A nontrivial divisor of  $n$  is a divisor  $m$  of  $n$  such that  $m \neq 1$  and  $m \neq n$ .

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**Definition 0.6.** Let  $n$  be a natural number.  $n$  is composite iff  $n > 1$  and  $n$  has a nontrivial divisor.

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**Proposition 0.7.** Let  $n$  be a natural number. Then  $n \mid 0$ .

*Proof.* We have  $n \cdot 0 = 0$ . Hence  $n \mid 0$ .  $\square$

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**Proposition 0.8.** Let  $n$  be a natural number. If  $0 \mid n$  then  $n = 0$ .

*Proof.* Assume  $0 \mid n$ . Consider a natural number  $k$  such that  $0 \cdot k = n$ . Then  $n = 0$ .  $\square$

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**Proposition 0.9.** Let  $n$  be a natural number. Then  $1 \mid n$ .

*Proof.* We have  $1 \cdot n = n$ . Hence  $1 \mid n$ .  $\square$

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**Proposition 0.10.** Let  $n$  be a natural number. Then  $n \mid n$ .

*Proof.* We have  $n \cdot 1 = n$ . Hence  $n \mid n$ .  $\square$

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**Proposition 0.11.** Let  $n$  be a natural number. If  $n \mid 1$  then  $n = 1$ .

*Proof.* Assume  $n \mid 1$ . Take a natural number  $k$  such that  $n \cdot k = 1$ . Suppose  $n \neq 1$ . Then  $n < 1$  or  $n > 1$ .

Case  $n < 1$ . Then  $n = 0$ . Hence  $0 = 0 \cdot k = n \cdot k = 1$ . Contradiction. End.

Case  $n > 1$ . We have  $k \neq 0$ . Indeed if  $k = 0$  then  $1 = n \cdot k = n \cdot 0 = 0$ . Hence  $k \geq 1$ . Take a positive natural number  $l$  such that  $n = 1 + l$ . Then  $1 < 1 + l = n = n \cdot 1 \leq n \cdot k$ . Hence  $1 < n$ . Contradiction. End.  $\square$

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**Proposition 0.12.** Let  $n, m, k$  be natural numbers. If  $n \mid m$  then  $n \mid m \cdot k$ .

*Proof.* Assume  $n \mid m$ . Take  $l \in \mathbb{N}$  such that  $n \cdot l = m$ . Then  $n \cdot (l \cdot k) = (n \cdot l) \cdot k = m \cdot k$ . Hence  $n \mid m \cdot k$ .  $\square$

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**Corollary 0.13.** Let  $n, m, k$  be natural numbers. If  $n \mid m$  then  $n \mid k \cdot m$ .

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**Proposition 0.14.** Let  $n, m, k$  be natural numbers. If  $n \mid m \mid k$  then  $n \mid k$ .

*Proof.* Assume  $n \mid m$  and  $m \mid k$ . Take natural numbers  $l, l'$  such that  $n \cdot l = m$  and  $m \cdot l' = k$ . Then  $n \cdot (l \cdot l') = (n \cdot l) \cdot l' = m \cdot l' = k$ . Hence  $n \mid k$ .  $\square$

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**Proposition 0.15.** Let  $n, m$  be natural numbers such that  $n \neq 0$ . If  $n \mid m$  and  $m \mid n$  then  $n = m$ .

*Proof.* Assume  $n \mid m$  and  $m \mid n$ . Take natural numbers  $k, k'$  such that  $n \cdot k = m$  and  $m \cdot k' = n$ . Then  $n = m \cdot k' = (n \cdot k) \cdot k' = n \cdot (k \cdot k')$ . Hence  $k \cdot k' = 1$ . Thus  $k = 1 = k'$ . Therefore  $n = m$ .  $\square$

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**Proposition 0.16.** Let  $n, m, k$  be natural numbers. If  $n \mid m$  then  $k \cdot n \mid k \cdot m$ .

*Proof.* Assume  $n \mid m$ . Take a natural number  $l$  such that  $n \cdot l = m$ . Then  $(k \cdot n) \cdot l = k \cdot (n \cdot l) = k \cdot m$ . Hence  $k \cdot n \mid k \cdot m$ .  $\square$

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**Proposition 0.17.** Let  $n, m, k$  be natural numbers. Assume  $k \neq 0$ . If  $k \cdot n \mid k \cdot m$  then  $n \mid m$ .

*Proof.* Assume  $k \cdot n \mid k \cdot m$ . Take a natural number  $l$  such that  $(k \cdot n) \cdot l = k \cdot m$ . Then  $k \cdot (n \cdot l) = k \cdot m$ . Hence  $n \cdot l = m$  (by ARITHMETIC\_06\_8575191374364672). Thus  $n \mid m$ .  $\square$

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**Proposition 0.18.** Let  $n, m, k$  be natural numbers. If  $k \mid n$  and  $k \mid m$  then  $k \mid (n' \cdot n) + (m' \cdot m)$  for all natural numbers  $n', m'$ .

*Proof.* Assume  $k \mid n$  and  $k \mid m$ . Let  $n', m'$  be natural numbers. Take natural numbers  $l, l'$  such that  $k \cdot l = n$  and  $k \cdot l' = m$ . Then

$$\begin{aligned}
 & k \cdot ((n' \cdot l) + (m' \cdot l')) \\
 &= (k \cdot (n' \cdot l)) + (k \cdot (m' \cdot l')) \\
 &= ((k \cdot n') \cdot l) + ((k \cdot m') \cdot l') \\
 &= (n' \cdot (k \cdot l)) + (m' \cdot (k \cdot l')) \\
 &= (n' \cdot n) + (m' \cdot m).
 \end{aligned}$$

□

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**Corollary 0.19.** Let  $n, m, k$  be natural numbers. If  $k \mid n$  and  $k \mid m$  then  $k \mid n + m$ .

*Proof.* Assume  $k \mid n$  and  $k \mid m$ . Take  $n' = 1$  and  $m' = 1$ . Then  $k \mid (n' \cdot n) + (m' \cdot m)$ .  $(n' \cdot n) + (m' \cdot m) = n + m$ . Hence  $k \mid n + m$ . □

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**Proposition 0.20.** Let  $n, m, k$  be natural numbers. If  $k \mid n$  and  $k \mid n + m$  then  $k \mid m$ .

*Proof.* Assume  $k \mid n$  and  $k \mid n + m$ .

Case  $k = 0$ . Obvious.

Case  $k \neq 0$ . Take a natural number  $l$  such that  $n = k \cdot l$ . Take a natural number  $l'$  such that  $n + m = k \cdot l'$ . Then  $(k \cdot l) + m = k \cdot l'$ . We have  $l' \geq l$ . Indeed if  $l' < l$  then  $n + m = k \cdot l' < k \cdot l = n$  (by ARITHMETIC\_06\_5048640368279552). Hence we can take a natural number  $l''$  such that  $l' = l + l''$ . Then  $(k \cdot l) + m = k \cdot l' = k \cdot (l + l'') = (k \cdot l) + (k \cdot l'')$ . Indeed  $k \cdot (l + l'') = (k \cdot l) + (k \cdot l'')$  (by ARITHMETIC\_06\_9001524774567936). Thus  $m = (k \cdot l'')$  (by ARITHMETIC\_03\_8445946379632640). Indeed  $k \cdot l$  and  $k \cdot l''$  are natural numbers. Therefore  $k \mid m$ . End. □

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**Proposition 0.21.** Let  $n, m$  be natural numbers such that  $n, m \neq 0$ . If  $m \mid n$  then  $m \leq n$ .

*Proof.* Assume  $m \mid n$ . Take a natural number  $k$  such that  $m \cdot k = n$ . If

$k = 0$  then  $n = m \cdot k = m \cdot 0 = 0$ . Thus  $k \geq 1$ . Assume  $m > n$ . Then  $n = m \cdot k \geq m \cdot 1 = m > n$ . Hence  $n > n$ . Contradiction.  $\square$