Cantor's Theorem

Naproche formalization:

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Theorem (Cantor's Theorem). Let x be a set. There exists no surjective map from x onto $\mathcal{P}(x)$.

Proof. Assume the contrary. Take a surjective map f from x onto $\mathcal{P}(x)$. Define $C = \{u \in x \mid u \notin f(u)\}$. Then $C \in \mathcal{P}(x)$. Hence we can take a $u \in x$ such that $f(u) = C$. Then $u \in C$ iff $u \in f(u)$ iff $u \notin C$. Contradiction. $\square$

Corollary. Let x be a set. Then $|x| < |\mathcal{P}(x)|$.

Proof. Case x is empty. Obvious.

Case x is nonempty. Assume the contrary. Then $|x| \geq |\mathcal{P}(x)|$. Hence there exists a surjective map from x onto $\mathcal{P}(x)$ (by `SET_THEORY_06_192336220913664`). Indeed $\mathcal{P}(x)$ is a nonempty set. Contradiction (by Cantor's Theorem). End. $\square$

Corollary. For every ordinal α there exists a cardinal greater than α.

Proof. Let α be an ordinal. Take $\kappa = |\mathcal{P}(\alpha)|$. Then $\kappa > |\alpha|$.

Let us show that $\kappa > \alpha$. Assume the contrary. Then $|\mathcal{P}(\alpha)| = \kappa \leq \alpha$. Hence $\kappa = |\mathcal{P}(\alpha)| = ||\mathcal{P}(\alpha)|| \leq |\alpha|$. Contradiction. End. $\square$

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