

# Finite and Infinite Classes

## Finite Classes

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**Definition 0.1.** Let  $X$  be a class and  $n$  be a natural number.  $X$  has  $n$  elements iff  $X$  is equinumerous to  $\{1, \dots, n\}$ .

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**Definition 0.2.** Let  $X$  be a class.  $X$  is finite iff there exists a natural number  $n$  such that  $X$  has  $n$  elements.

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**Proposition 0.3.** Let  $X, Y$  be classes. If  $X$  is finite and  $Y$  is equinumerous to  $X$  then  $Y$  is finite.

*Proof.* Assume that  $X$  is finite and  $Y$  is equinumerous to  $X$ . Take a natural number  $n$  and a bijection  $f$  between  $\{1, \dots, n\}$  and  $X$  and a bijection  $g$  between  $X$  and  $Y$ . Then  $g \circ f$  is a bijection between  $\{1, \dots, n\}$  and  $Y$  (by FOUNDATIONS\_08\_6435206693126144). Indeed  $X, Y$  are classes. Hence  $Y$  is finite.  $\square$

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**Proposition 0.4.** Let  $X$  be a class.  $X$  has 0 elements iff  $X = \emptyset$ .

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**Proposition 0.5.** Let  $X$  be a class.  $X$  has 1 element iff  $X = \{a\}$  for some object  $a$ .

*Proof.* Case  $X$  has 1 element. Take a bijection  $f$  between  $\{1, \dots, 1\}$  and  $X$ . We have  $\{1, \dots, 1\} = \{1\}$ . Hence  $X = \{f(1)\}$ . End.

Case  $X = \{a\}$  for some object  $a$ . Consider an object  $a$  such that  $X = \{a\}$ . Define  $f(x) = 1$  for  $x \in \{a\}$ . Then  $f$  is a bijection between  $\{a\}$  and  $\{1, \dots, 1\}$ . End.  $\square$

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**Proposition 0.6.** Let  $X$  be a class.  $X$  has 2 elements iff  $X = \{a, b\}$  for some distinct objects  $a, b$ .

*Proof.* Case  $X$  has 2 elements. Take a bijection  $f$  between  $\{1, \dots, 2\}$  and  $X$ . We have  $\{1, \dots, 2\} = \{1, 2\}$ . Hence  $X = \{f(1), f(2)\}$ . We have  $f(1) \neq f(2)$ . End.

Case  $X = \{a, b\}$  for some distinct objects  $a, b$ . Consider distinct objects  $a, b$  such that  $X = \{a, b\}$ . Define

$$f(x) = \begin{cases} 1 & : x = a \\ 2 & : x = b \end{cases}$$

for  $x \in \{a, b\}$ . Then  $f$  is a bijection between  $\{a, b\}$  and  $\{1, \dots, 2\}$ . Indeed  $f$  is injective and surjective onto  $\{1, \dots, 2\}$ . End.  $\square$

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**Proposition 0.7.** Let  $n$  be a natural number and  $X$  be a class that has  $n$  elements and  $a$  be an object such that  $a \notin X$ . Then  $X \cup \{a\}$  has  $n + 1$  elements.

*Proof.* Take a bijection  $f$  between  $X$  and  $\{1, \dots, n\}$ . Define

$$g(x) = \begin{cases} f(x) & : x \in X \\ n + 1 & : x = a \end{cases}$$

for  $x \in X \cup \{a\}$ .

(1)  $g$  is a map from  $X \cup \{a\}$  to  $\{1, \dots, n + 1\}$ .

Indeed we can show that  $g(x) \in \{1, \dots, n + 1\}$  for all  $x \in X \cup \{a\}$ . Let  $x \in X \cup \{a\}$ . If  $x \in X$  then  $g(x) \in \{1, \dots, n\}$ . If  $x = a$  then  $g(x) = n + 1$ . End.

(2)  $g$  is injective.

(3)  $g$  is surjective onto  $\{1, \dots, n + 1\}$ .

*Proof.* Let  $k \in \{1, \dots, n + 1\}$ .

Case  $k \leq n$ . Then  $k \in \{1, \dots, n\}$ . Hence we can take a  $x \in X$  such that  $k = f(x)$ . End.

Case  $k = n + 1$ . Then  $k = g(a)$ . End. Qed.

Hence  $g$  is a bijection between  $X \cup \{a\}$  and  $\{1, \dots, n + 1\}$ .  $\square$

## Infinite Classes

**Definition 0.8.** Let  $X$  be a class.  $X$  is infinite iff  $X$  is not finite.

**Proposition 0.9.** Let  $X, Y$  be classes. If  $X$  is infinite and  $Y$  is equinumerous to  $X$  then  $Y$  is infinite.

*Proof.* Assume that  $Y$  is equinumerous to  $X$ . If  $Y$  is finite then  $X$  is finite. Hence if  $X$  is infinite then  $Y$  is infinite.  $\square$