Maps

Range

FOUNDATIONS_06_4284980337311744

Definition 0.1. Let f be a map. A value of f is an object b such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$.

FOUNDATIONS_06_1938831225913344

Definition 0.2. Let f be a map. The range of f is $\{f(a) \mid a \in \mathrm{dom}(f)\}$.

Let $\mathrm{range}(f)$ stand for the range of f.

FOUNDATIONS_06_6386349418479616

Proposition 0.3. Let f be a map and b be an object. b is a value of f iff $b \in \mathrm{range}(f)$.

Proof. Case b is a value of f. Take $a \in \mathrm{dom}(f)$ such that $b = f(a)$. b is an object. Hence $b \in \mathrm{range}(f)$. End.

Case $b \in \mathrm{range}(f)$. Then b is an object such that $b = f(a)$ for some $a \in \mathrm{dom}(f)$. Hence b is a value of f. End. $\square$

Identity Map

FOUNDATIONS_06_1920902360989696

Definition 0.4. Let A be a class. id_A is the map h such that h is defined on A and $h(a) = a$ for all $a \in A$.

Let the identity map on A stand for id_A.

Composition

FOUNDATIONS_06_7605717729017856

Definition 0.5. Let f, g be maps. Assume $\mathrm{range}(f) \subseteq \mathrm{dom}(g)$. $g \circ f$ is the map h such that h is defined on $\mathrm{dom}(f)$ and $h(a) = g(f(a))$ for all $a \in \mathrm{dom}(f)$.

Let the composition of g and f stand for $g \circ f$.

Restriction

FOUNDATIONS_06_7095412741636096

Definition 0.6. Let f be a map and $X \subseteq \mathrm{dom}(f)$. $f \upharpoonright X$ is the map h such that h is defined on X and $h(a) = f(a)$ for all $a \in X$.

Let the restriction of f to X stand for $f \upharpoonright X$.

FOUNDATIONS_06_2170189258948608

Proposition 0.7. Let A be a class and $X \subseteq A$. Then $\mathrm{id}_A \upharpoonright X = \mathrm{id}_X$.

Image and Preimage

FOUNDATIONS_06_3038237683613696

Definition 0.8. Let f be a map and A be a class. The image of A under f is $\{f(a) \mid a \in \mathrm{dom}(f) \cap A\}$.

Let the direct image of A under f stand for the image of A under f. Let $f_*(A)$ stand for the image of A under f.

Let $f[A]$ stand for $f_*(A)$.

FOUNDATIONS_06_4563167805964288

Definition 0.9. Let f be a map and B be a class. The preimage of B under f is $\{a \in \mathrm{dom}(f) \mid f(a) \in B\}$.

Let the inverse image of B under f stand for the preimage of B under f. Let $f^*(B)$ stand for the preimage of B under f.

Maps Between Classes

FOUNDATIONS_06_6934038600220672

Definition 0.10. Let A be a class. A map of A is a map f such that $\mathrm{dom}(f) = A$.

Let a function of A stand for a map of A that is a function.

FOUNDATIONS_06_7725375157174272

Definition 0.11. Let B be a class. A map to B is a map f such that $f(a) \in B$ for each $a \in \mathrm{dom}(f)$.

Let a function to B stand for a map to B that is a function.

FOUNDATIONS_06_2823507398361088

Definition 0.12. Let A, B be classes. A map from A to B is a map f such that $\mathrm{dom}(f) = A$ and $f(a) \in B$ for each $a \in A$.

Let $f : A \to B$ stand for f is a map from A to B.

Let a function from A to B stand for a map from A to B that is a function.

FOUNDATIONS_06_3390734908522496

Definition 0.13. Let A be a class. A map on A is a map from A to A.

Let a function on A stand for a map on A that is a function.

FOUNDATIONS_06_3312973569327104

Proposition 0.14. Let A, B be classes and $f, g : A \to B$. Assume that $f(a) = g(a)$ for all $a \in A$. Then $f = g$.

Proposition 0.15. Let A, B be classes and f be a map of A. Assume that $f(a) \in B$ for all $a \in A$. Then f is a map from A to B iff $\mathrm{range}(f) \subseteq B$.

FOUNDATIONS_06_5104361690628096

Proposition 0.16. Let A be a class. Then id_A is a map on A.

FOUNDATIONS_06_1706446651654144

Proposition 0.17. Let A, B, C be classes and $f : A \to B$ and $g : B \to C$. Then $g \circ f : A \to C$.

FOUNDATIONS_06_4078561256275968

Proposition 0.18. Let A, B be classes and $f : A \to B$ and $X \subseteq A$. Then $f \restriction X : X \to B$.

FOUNDATIONS_06_3964401904254976

Proposition 0.19. Let A, B be classes and $f : A \to B$. Then $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$.

Proof. A is the domain of $f \circ \mathrm{id}_A$ and the domain of f and the domain of $\mathrm{id}_B \circ f$. We have $(f \circ \mathrm{id}_A)(a) = f(\mathrm{id}_A(a)) = f(a) = \mathrm{id}_B(f(a)) = (\mathrm{id}_B \circ f)(a)$ for all $a \in A$. Hence $f \circ \mathrm{id}_A = f = \mathrm{id}_B \circ f$. □

FOUNDATIONS_06_3118771061391360

Proposition 0.20. Let A be a class and $X \subseteq A$. Then $\mathrm{id}_A \restriction X = \mathrm{id}_X$.

Proof. We have $\mathrm{dom}(\mathrm{id}_A \restriction X) = X = \mathrm{dom}(\mathrm{id}_X)$. $(\mathrm{id}_A \restriction X)(a) = \mathrm{id}_A(a) = a = \mathrm{id}_X(a)$ for all $a \in X$. Hence $\mathrm{id}_A \restriction X = \mathrm{id}_X$. □

FOUNDATIONS_06_6866147389472768

Proposition 0.21. Let A, B, C, D be classes and $f : A \to B$ and $g : B \to C$ and $h : C \to D$. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof. $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are maps from A to D.

Let us show that $(h \circ (g \circ f))(a) = ((h \circ g) \circ f)(a)$ for all $a \in A$. Let $a \in A$. Then $(h \circ (g \circ f))(a) = h((g \circ f)(a)) = h(g(f(a))) = (h \circ g)(f(a)) = ((h \circ g) \circ f)(a)$. End.

Hence $h \circ (g \circ f) = (h \circ g) \circ f$. □

Classes of Functions

FOUNDATIONS_06_5119110467813376

Definition 0.22. Let A, B be classes. $[A \to B]$ is the class of all functions from A to B.

Fixed Points

FOUNDATIONS_06_2177076576649216

Definition 0.23. Let f be a map. A fixed point of f is an element x of $\mathrm{dom}(f)$ such that $f(x) = x$.