

Cardinal Numbers

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Definition 0.1. Let x be a set. The cardinality of x is the ordinal κ such that κ is equinumerous to x and every ordinal that is equinumerous to x is greater than or equal to κ .

Let $|x|$ stand for the cardinality of x .

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Definition 0.2. A cardinal number is an ordinal κ such that $\kappa = |x|$ for some set x .

Let a cardinal stand for a cardinal number.

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Proposition 0.3. Let κ be a cardinal. Then $|\kappa| = \kappa$.

Proof. κ is an ordinal that is equinumerous to κ . Hence $|\kappa| \leq \kappa$. Consider a set x such that $\kappa = |x|$. Then $|\kappa|$ is an ordinal that is equinumerous to x . Hence $\kappa \leq |\kappa|$. Thus $|\kappa| = \kappa$. \square

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Proposition 0.4. Let x, y be sets. Then x and y are equinumerous iff $|x| = |y|$.

Proof. Case x and y are equinumerous. Take a bijection f between x and y . Consider a bijection g between y and $|y|$. Then $g \circ f$ is a bijection between x and $|y|$ (by FOUNDATIONS_08_6435206693126144). Hence x and $|y|$ are equinumerous. Thus $|y| \geq |x|$.

f^{-1} is a bijection between y and x . Consider a bijection h between x and $|x|$. Then $h \circ f^{-1}$ is a bijection between y and $|x|$ (by FOUNDATIONS_08_6435206693126144). Hence y and $|x|$ are equinumerous. Thus $|x| \geq |y|$.

Therefore $|x| = |y|$. End.

Case $|x| = |y|$. Consider a bijection f between x and $|x|$ and a bijection g between $|y|$ and y . Then $g \circ f$ is a bijection between x and y . Hence x and y are equinumerous. End. \square