ZFC

FOUNDATIONS_10_5891530432708608

Proposition 0.1. $\emptyset$ is a set.

Proof. Take a set x (by FOUNDATIONS_10_2362039748001792). Define $A = \{y \in x \mid y \neq y\}$. Then A is a set (by FOUNDATIONS_10_2263707272871936). We have $A = \emptyset$. Hence $\emptyset$ is a set. $\square$

FOUNDATIONS_10_4052198354845696

Proposition 0.2. Let x, y be sets. Then $x \cup y$ is a set.

Proof. Take $X = \{x, y\}$. Then X is a set. Hence $\bigcup X$ is a set (by FOUNDATIONS_10_5536459412996096). Indeed X is a system of sets. We have $x \cup y = \bigcup X$. Thus $x \cup y$ is a set. $\square$

FOUNDATIONS_10_4475839687163904

Proposition 0.3. Let x, y be sets. Then $x \cap y$ is a set.

Proof. We have $x \cap y \subseteq x$. Hence $x \cap y$ is a set (by FOUNDATIONS_10_2263707272871936). $\square$

FOUNDATIONS_10_7795203882614784

Proposition 0.4. Let x, y be sets. Then $x \setminus y$ is a set.

Proof. We have $x \setminus y \subseteq x$. Hence $x \setminus y$ is a set (by FOUNDATIONS_10_2263707272871936). $\square$

FOUNDATIONS_10_4458706448154624

Proposition 0.5. Let x, y be sets. Then $x \times y$ is a set.

Proof. $\{a\}$ and $\{a, b\}$ are sets for each $a \in x$ and each $b \in y$. Define $P = \{\{\{a\}, \{a, b\}\} \mid a \in x \text{ and } b \in y\}$.

(1) P is a set.
Proof. Let us show that $P \subseteq \mathcal{P}(\mathcal{P}(x \cup y))$. Let $p \in P$. Consider $a \in x$ and $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$. Then $a, b \in x \cup y$. Hence $\{a\}, \{a, b\} \in \mathcal{P}(x \cup y)$. Thus $\{\{a\}, \{a, b\}\} \in \mathcal{P}(\mathcal{P}(x \cup y))$. End.

$x \cup y$ is a set. Consequently $\mathcal{P}(\mathcal{P}(x \cup y))$ is a set (by FOUNDATIONS_10_5862230203564032). Therefore P is a set (by FOUNDATIONS_10_22637072728

`71936`). Qed.

Define $l(p)$ = "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in a" for $p \in P$. Define $r(p)$ = "choose $a \in x$, choose $b \in y$ such that $p = \{\{a\}, \{a, b\}\}$ in b" for $p \in P$.

Define $f(p) = (l(p), r(p))$ for $p \in P$.

Let us show that for any objects u, u', v, v' if $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$ then $u = u'$ and $v = v'$. Let u, u', v, v' be objects. Assume $\{\{u\}, \{u, v\}\} = \{\{u'\}, \{u', v'\}\}$. Then ($\{u\} = \{u'\}$ or $\{u\} = \{u', v'\}$) and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). Thus ($\{u\} = \{u'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$)) or ($\{u\} = \{u', v'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$)).

Case $\{u\} = \{u'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). We have $\{u\} = \{u'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = u' = v$. Hence $\{\{u\}, \{u, u\}\} = \{\{u\}, \{u, v'\}\}$ (by 1). Thus $\{\{u\}\} = \{\{u\}, \{u, v'\}\}$. Therefore $\{u\} = \{u, v'\}$. Consequently $v' = u = v$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End.

Case $\{u\} = \{u', v'\}$ and ($\{u, v\} = \{u'\}$ or $\{u, v\} = \{u', v'\}$). We have $\{u\} = \{u', v'\}$. Hence $u = u'$.

Case $\{u, v\} = \{u'\}$. Then $u = v = u'$. Hence $v = v'$. End.

Case $\{u, v\} = \{u', v'\}$. Then $\{u, v\} = \{u, v'\}$. Hence $v = v'$. End. End. End.

$\{\{a\}, \{a, b\}\} \in \mathrm{dom}(f)$ for any $a \in x$ and any $b \in y$.
Proof. Let $a \in x$ and $b \in y$. Then $\{\{a\}, \{a, b\}\} \in P$. Qed.

Let us show that for any $a \in x$ and any $b \in y$ we have $f(\{\{a\}, \{a, b\}\}) = (a, b)$. Let $a \in x$ and $b \in y$. Take $p = \{\{a\}, \{a, b\}\}$. Then p is a set. Then we can choose $a' \in x$ and $b' \in y$ such that $p = \{\{a'\}, \{a', b'\}\}$ and $l(p) = a'$. Then $a = a'$ and $b = b'$. Hence $l(p) = a$. Choose $a'' \in x$ and $b'' \in y$ such that $p = \{\{a''\}, \{a'', b''\}\}$ and $r(p) = b''$. Then $a = a''$ and $b = b''$. Thus $r(p) = b$. Therefore $f(p) = (a, b)$. End.

(2) $x \times y = f[P]$.
Proof. For all $p \in P$ we have $l(p) \in x$ and $r(p) \in y$. Hence $f(p) \in x \times y$ for all $p \in P$. Therefore $f[P] \subseteq x \times y$.

Let us show that $x \times y \subseteq f[P]$. Let $z \in x \times y$. Take $a \in x$ and $b \in y$ such that $z = (a, b)$. Then $(a, b) = f(\{\{a\}, \{a, b\}\})$. Hence there exists a $p \in P$ such that $(a, b) = f(p)$. Thus $(a, b) \in f[P]$. End.

Consequently $x \times y = f[P]$. Qed.

Thus $x \times y$ is the image of some set under some map. Therefore $x \times y$ is a set (by `FOUNDATIONS_10_8142956584239104`). □

FOUNDATIONS_10_5486815207227392

Proposition 0.6. Let X be a nonempty system of sets. Then $\bigcap X$ is a set.

Proof. Take an element x of X. Then $\bigcap X \subseteq x$. Hence $\bigcap X$ is a set (by FOUNDATIONS_10_2263707272871936). ☐

FOUNDATIONS_10_7598384349184000

Proposition 0.7. Let f be a map such that $\mathrm{dom}(f)$ is a set. Then $\mathrm{range}(f)$ is a set.

Proof. $\mathrm{range}(f) = f_*(\mathrm{dom}(f))$ and $f_*(\mathrm{dom}(f))$ is a set. Hence $\mathrm{range}(f)$ is a set (by FOUNDATIONS_10_8142956584239104). ☐

FOUNDATIONS_10_8631339572002816

Proposition 0.8. Let A be a class and x be a set. Assume that there exists an injective map from A to x. Then A is a set.

Proof. Consider an injective map f from A to x. Then f^{-1} is a bijection between $\mathrm{range}(f)$ and A. $\mathrm{range}(f)$ is a set and A is the image of $\mathrm{range}(f)$ under f^{-1}. Indeed $\mathrm{range}(f) \subseteq x$. Thus A is a set (by FOUNDATIONS_10_8142956584239104). ☐

FOUNDATIONS_10_8812282138066944

Proposition 0.9. There exist no sets x, y such that $x \in y$ and $y \in x$.

Proof. Assume the contrary. Take sets x, y such that $x \in y$ and $y \in x$. Consider an element z of $\{x, y\}$ such that $\{x, y\}$ and z are disjoint (by FOUNDATIONS_10_1320008569323520). Indeed $\{x, y\}$ is a nonempty system of sets. Then we have $z = x$ or $z = y$.

Case $z = x$. Then x and $\{x, y\}$ are disjoint. Hence $y \notin x$. Contradiction. End.

Case $z = y$. Then y and $\{x, y\}$ are disjoint. Hence $x \notin y$. Contradiction. End. ☐

FOUNDATIONS_10_3086917813927936

Corollary 0.10. Let x be a set. Then $x \notin x$.

FOUNDATIONS_10_4589652321021547

Proposition 0.11. Let f, g be functions. Assume that $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $f(a) = g(a)$ for all $a \in \mathrm{dom}(f)$. Then $f = g$.

FOUNDATIONS_10_4105036244189184

Proposition 0.12. Let x, y be sets. Then $[x \to y]$ is a set.

Proof. Define $R = \{F \in \mathcal{P}(x \times y) \mid$ (for all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$) and for all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'\}$.

[prover vampire] Every element of R is a set. Define $h(F) = \lambda a \in x$. "choose $b \in y$ such that $(a, b) \in F$ in b" for $F \in R$. [prover eprover]

Let us show that $[x \to y] \subseteq \mathrm{range}(h)$. Let $f \in [x \to y]$. Define $F = \{(a, f(a)) \mid a \in x\}$.

Then $F \in R$.
Proof. Define $g(a) = (a, f(a))$ for $a \in x$. Then $F = \mathrm{range}(g)$. Hence F is a set. Thus $F \in \mathcal{P}(x \times y)$. Indeed $F \subseteq x \times y$.

(1) For all $a \in x$ there exists a $b \in y$ such that $(a, b) \in F$.

(2) For all $a \in x$ and all $b, b' \in y$ such that $(a, b), (a, b') \in F$ we have $b = b'$.

[prover vampire] Hence the thesis. [prover eprover] End.

$\mathrm{dom}(f) = \mathrm{dom}(h(F))$ and for each $a \in \mathrm{dom}(f)$ we have $h(F)(a) = f(a)$. Hence $f = h(F)$. Thus $f \in \mathrm{range}(h)$. End.

Therefore $[x \to y]$ is a set. Indeed R is a set. □