

Factorial and Exponentiation

ARITHMETIC_12_8113296594960384

Proposition 0.1. Let n be a natural number. Assume $n \geq 2$. Then $n^n > n!$.

Proof. (a) Define $\Phi = \{n' \in \mathbb{N}_{\geq 2} \mid n'^{n'} > n'!\}$.

(1) Φ contains 2.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 2$.

Take $X = (n' + 1)^{n'} \cdot (n' + 1)$ and $Y = n'^{n'} \cdot (n' + 1)$. Then $X > Y$.

Proof. We have $n' + 1 > n'$ and $n' \neq 0$. Thus $(n' + 1)^{n'} > n'^{n'}$. $n' + 1$ is nonzero. Hence $(n' + 1)^{n'} \cdot (n' + 1) > n'^{n'} \cdot (n' + 1)$. Qed.

Take $Z = n'! \cdot (n' + 1)$. Then $Y > Z$. Indeed $n'^{n'} > n'!$ and $n' + 1 \neq 0$.

Hence $(n' + 1)^{n'+1} = X > Y > Z = (n' + 1)!$. Thus $(n' + 1)^{n'+1} > (n' + 1)!$. Qed.

Therefore Φ contains every element of $\mathbb{N}_{\geq 2}$ (by ARITHMETIC_04_49765992 69113856). Consequently $n^n > n!$ (by a). \square

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Proposition 0.2. Let n be a natural number. Assume $n \geq 4$. Then $n! > 2^n$.

Proof. (a) Define $\Phi = \{n' \in \mathbb{N}_{\geq 4} \mid n'! > 2^{n'}\}$.

(1) Φ contains 4.

Proof.

$$\begin{aligned}
 & 4! \\
 &= 4 \cdot (3 \cdot 2) \\
 &= 2 \cdot (2 \cdot (3 \cdot 2)) \\
 &= 3 \cdot (2 \cdot (2 \cdot 2)) \\
 &> 2 \cdot (2 \cdot (2 \cdot 2)) \\
 &= 2^4.
 \end{aligned}$$

Qed.

(2) For all $n' \in \Phi$ we have $n' + 1 \in \Phi$.

Proof. Let $n' \in \Phi$. Then $n' \geq 4$ and $n'! > 2^{n'}$.

Take $X = n'! \cdot (n' + 1)$ and $Y = 2^{n'} \cdot (n' + 1)$. Then $X > Y$.

Take $Z = 2^{n'} \cdot 2$. Then $Y > Z$. Indeed $n' + 1 > 2$.

Hence $(n' + 1)! = X > Y > Z = 2^{n'+1}$. Thus $(n' + 1)! > 2^{n'+1}$. Qed.

Therefore Φ contains every element of $\mathbb{N}_{\geq 4}$ (by ARITHMETIC_04_49765992 69113856). Consequently $n! > 2^n$ (by a). \square