

Infinite Cardinal Numbers

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Definition 0.1. **Card** is the collection of all infinite cardinals.

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Proposition 0.2. Let α be an infinite ordinal. Then $|\text{succ}(\alpha)| = |\alpha|$.

Proof. For any $\beta \in \text{succ}(\alpha)$ we have $\beta < \omega$ or $\omega \leq \beta < \alpha$ or $\beta = \alpha$. Define

$$f(\beta) = \begin{cases} \text{succ}(\beta) & : \beta < \omega \\ \beta & : \omega \leq \beta < \alpha \\ 0 & : \beta = \alpha \end{cases}$$

for $\beta \in \text{succ}(\alpha)$.

Then f is a map from $\text{succ}(\alpha)$ to α . Indeed we can show that $f(\beta) \in \alpha$ for all $\beta \in \text{succ}(\alpha)$.

Proof. Let $\beta \in \text{succ}(\alpha)$.

Case $\beta < \omega$. Then $f(\beta) = \text{succ}(\beta) < \omega \leq \alpha$. End.

Case $\omega \leq \beta < \alpha$. Then $f(\beta) = \beta < \alpha$. End.

Case $\beta = \alpha$. Then $f(\beta) = 0 < \alpha$. End. Qed.

f is surjective onto α . Indeed we can show that for any $\beta \in \alpha$ there exists a $\gamma \in \text{succ}(\alpha)$ such that $\beta = f(\gamma)$.

Proof. Let $\beta \in \alpha$. Then $\beta = 0$ or $0 < \beta < \omega$ or $\beta \geq \omega$.

Case $\beta = 0$. Then $\beta = f(\alpha)$. End.

Case $0 < \beta < \omega$. Take an ordinal β' such that $\beta = \text{succ}(\beta')$. Then $\beta' < \omega$. Hence $\beta = f(\beta')$. End.

Case $\beta \geq \omega$. Then $\beta = f(\beta)$. End. Qed.

f is injective. Indeed we can show that for all $\beta, \gamma \in \text{succ}(\alpha)$ if $\beta \neq \gamma$ then $f(\beta) \neq f(\gamma)$.

Proof. Let $\beta, \gamma \in \text{succ}(\alpha)$. Assume $\beta \neq \gamma$.

Case $\beta < \omega$. If $\gamma = \alpha$ then $f(\beta) = \text{succ}(\beta) \neq 0 = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = \text{succ}(\beta) < \omega \leq \gamma = f(\gamma)$. End.

Case $\omega \leq \beta < \alpha$. If $\gamma = \alpha$ then $f(\beta) = \beta \geq \omega > 0 = f(\gamma)$. If $\gamma < \omega$ then $f(\beta) = \beta \geq \omega > \text{succ}(\gamma) = f(\gamma)$. End.

Case $\beta = \alpha$. If $\gamma < \omega$ then $f(\beta) = 0 \neq \text{succ}(\gamma) = f(\gamma)$. If $\omega \leq \gamma < \alpha$ then $f(\beta) = 0 < \omega \leq \gamma = f(\gamma)$. End. Qed.

Hence f is a bijection between $\text{succ}(\alpha)$ and α . Therefore $\text{succ}(\alpha)$ and α are equinumerous. Consequently $|\text{succ}(\alpha)| = |\alpha|$. \square

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Proposition 0.3. Every infinite cardinal is a limit ordinal.

Proof. Let κ be an infinite cardinal. Suppose that κ is not a limit ordinal. $\kappa \neq 0$. Hence κ is a successor ordinal. Indeed κ is an ordinal. Thus we can take an ordinal α such that $\kappa = \text{succ}(\alpha)$. We have $\alpha > \kappa \geq \omega$. Hence $|\text{succ}(\alpha)| = |\alpha|$. Thus $\alpha < |\kappa|$ and κ is equinumerous to κ . Contradiction. \square