

# König's Theorem

Naproche formalization:

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2018 - 2021

König's Theorem is an important set-theoretical result about the arithmetic of cardinals. It was proved by Julius König in 1905 [Koenig1905]. The proof is reminiscent of Cantor's diagonal argument for proving that  $\kappa < 2^\kappa$ .

## Sums and Products of cardinals

Let  $f_i$  stand for  $f(i)$ . Let  $D$  denote a set.

**Definition.** A sequence of cardinals on  $D$  is a function  $\kappa$  such that  $\text{dom}(\kappa) = D$  and  $\kappa_i$  is a cardinal for every element  $i$  of  $D$ .

**Definition.** Let  $\kappa$  be a sequence of cardinals on  $D$ .

$$\bigsqcup_{i \in D} \kappa_i = \{(n, i) \mid i \text{ is an element of } D \text{ and } n \text{ is an element of } \kappa_i\}.$$

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$$\prod_{i \in D} \kappa_i = \left\{ f \mid f \text{ is a function and } \text{dom}(f) = D \text{ and } f(i) \text{ is an element of } \kappa_i \text{ for every element } i \text{ of } D \right\}.$$

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König's Theorem requires some form of the axiom of choice. Currently choice is built into Naproche by the *choose* construct in function definitions. The axiom of choice is also required to show that products of non-empty factors are themselves non-empty:

**Lemma (Choice).** Let  $\lambda$  be a sequence of cardinals on  $D$ . Assume that  $\lambda_i$  has an element for every element  $i$  of  $D$ . Then  $\prod_{i \in D} \lambda_i$  has an element. *Proof.* Define  $f(i) =$  "choose an element  $v$  of  $\lambda_i$  in  $v$ " for  $i$  in  $D$ . Then  $f$  is an element of  $\prod_{i \in D} \lambda_i$ .  $\square$

## König's theorem

**Theorem (König).** Let  $\kappa, \lambda$  be sequences of cardinals on  $D$ . Assume that for every element  $i$  of  $D$   $\kappa_i < \lambda_i$ . Then

$$\sum_{i \in D} \kappa_i < \prod_{i \in D} \lambda_i.$$

*Proof by contradiction.* Assume the contrary. Then

$$\prod_{i \in D} \lambda_i \leq \sum_{i \in D} \kappa_i.$$

Take a surjective map  $G$  from  $\bigsqcup_{i \in D} \kappa_i$  to  $\prod_{i \in D} \lambda_i$ . Indeed  $\prod_{i \in D} \lambda_i$  and  $\sum_{i \in D} \kappa_i$  are nonempty sets. Take  $\Lambda = \bigcup \text{range}(G)$ . Then  $\Lambda$  is a set. Indeed  $\text{range}(G)$  is a set.

$(n, i) \in \text{dom}(G)$  for every  $i \in D$  and every  $n \in \kappa_i$ .  $G(n, i) \in \prod_{i \in D} \lambda_i$  for every  $i \in D$  and every  $n \in \kappa_i$ . Hence for every  $i \in D$  and every  $n \in \kappa_i$   $G(n, i)$  is a map such that  $i \in \text{dom}(G(n, i))$ .

Define  $\Delta(i) = \{G(n, i)(i) \in \Lambda \mid n \in \kappa_i\}$  for  $i \in D$ .

For every element  $f$  of  $\prod_{i \in D} \lambda_i$  and every element  $i$  of  $D$  we have  $f(i) \in \Lambda$ .

For every element  $i$  of  $D$  we have  $|\Delta(i)| < \lambda_i$ .

*Proof.* Let  $i$  be an element of  $D$ . Define  $F(n) = G(n, i)(i)$  for  $n$  in  $\kappa_i$ . Then  $F$  is a map from  $\kappa_i$  to  $\lambda_i$ . We have  $\Delta(i) = \{F(n) \mid n \in \kappa_i\}$ . Thus  $F[\kappa_i] = \Delta(i)$ . Therefore  $|\Delta(i)| = |F[\kappa_i]| \leq |\kappa_i| = \kappa_i < \lambda_i$ . Indeed  $|F[\kappa_i]| \leq |\kappa_i|$  (by SET\_THEORY\_06\_8113916590686208). Indeed  $\kappa_i$  and  $\lambda_i$  are sets. End.

Define  $f(i) =$  “choose an element  $v$  of  $\lambda_i \setminus \Delta(i)$  in  $v$ ” for  $i \in D$ . Indeed  $\lambda_i \setminus \Delta(i)$  is nonempty for each  $i \in D$ . Then  $f$  is an element of  $\prod_{i \in D} \lambda_i$ . Take an element  $j$  of  $D$  and an element  $m$  of  $\kappa_j$  such that  $G(m, j) = f$ .  $G(m, j)(j)$  is an element of  $\Delta(j)$  and  $f(j)$  is not an element of  $\Delta(j)$ . Contradiction.  $\square$