## Systems of Sets

FOUNDATIONS_10_5805323570905088

**Definition 0.1.** A system of sets is a class $X$ such that every element of $X$ is a set.

FOUNDATIONS_10_1631952387964928

**Definition 0.2.** A system of nonempty sets is a class $X$ such that every element of $X$ is a nonempty set.

FOUNDATIONS_10_943381479948288

**Definition 0.3.** Let $A$ be a class. A system of subsets of $A$ is a class $X$ such that every element of $X$ is a subset of $A$.

FOUNDATIONS_10_8268633648136192

**Proposition 0.4.** Let $A$ be a class. Then $\emptyset$ is a system of subsets of $A$.

FOUNDATIONS_10_7546016869908480

**Proposition 0.5.** Let $A$ be a class. Then $\mathcal{P}(A)$ is a system of subsets of $A$.

**Proposition 0.6.** Let $X, Y$ be systems of sets. Then $X \cup Y$ is a system of sets.

**Proposition 0.7.** Let $X, Y$ be systems of sets. Then $X \cap Y$ is a system of sets.

**Proposition 0.8.** Let $X, Y$ be systems of sets. Then $X \setminus Y$ is a system of sets.

## Unions Over Systems of Sets

FOUNDATIONS_10_541772562300928

**Definition 0.9.** Let $X$ be a system of sets. The union over $X$ is $\{a \mid a \in x \text{ for some } x \in X\}$.

Let $\bigcup X$ stand for the union over $X$.

FOUNDATIONS_10_4872701241982976

**Proposition 0.10.** $\bigcup \emptyset = \emptyset$.

*Proof.* $\bigcup \emptyset = \{a \mid a \in x \text{ for some } x \in \emptyset\}$. $\emptyset$ has no elements. Hence there is no object $a$ such that $a \in x$ for some $x \in \emptyset$. Thus $\bigcup \emptyset = \emptyset$. □

FOUNDATIONS_10_2559541585641472

**Proposition 0.11.** Let $x, y$ be sets. Then $\bigcup\{x, y\} = x \cup y$.

*Proof.* Let us show that $\bigcup\{x, y\} \subseteq x \cup y$. Let $a \in \bigcup\{x, y\}$. Then $a$ is contained in some element of $\{x, y\}$. Hence $a \in x$ or $a \in y$. Thus $a \in x \cup y$. End.

Let us show that $x \cup y \subseteq \bigcup\{x, y\}$. Let $a \in x \cup y$. Then $a \in x$ or $a \in y$. Hence $a$ is contained in some element of $\{x, y\}$. Therefore $a \in \bigcup\{x, y\}$. End. □

FOUNDATIONS_10_2157223832715264

**Corollary 0.12.** Let $x$ be a set. Then $\bigcup\{x\} = x$.

## Intersections Over Systems of Sets

FOUNDATIONS_10_2659345095458816

**Definition 0.13.** Let $X$ be a system of sets. The intersection over $X$ is $\{a \mid a \in x \text{ for all } x \in X\}$.

Let $\bigcap X$ stand for the intersection over $X$.

FOUNDATIONS_10_2809770322952192

**Proposition 0.14.** $\bigcap \emptyset$ is the class of all objects.

*Proof.* Define $V = \{x \mid x \text{ is an object }\}$. We have $\bigcap \emptyset \subseteq V$. Indeed every element of $\bigcap \emptyset$ is an object.

Let us show that $V \subseteq \bigcap \emptyset$. Let $a \in V$. Then $a$ is an object. For every $x \in \emptyset$ we have $a \in x$. Indeed $\emptyset$ has no elements. Thus $a \in \bigcap \emptyset$. End. $\square$

FOUNDATIONS_10_7851827447988224

**Proposition 0.15.** Let $x, y$ be sets. Then $\bigcap \{x, y\} = x \cap y$.

*Proof.* Let us show that $\bigcap \{x, y\} \subseteq x \cap y$. Let $a \in \bigcap \{x, y\}$. Then $a$ is contained in every element of $\{x, y\}$. Hence $a \in x$ and $a \in y$. Thus $a \in x \cap y$. End.

Let us show that $x \cap y \subseteq \bigcap \{x, y\}$. Let $a \in x \cap y$. Then $a \in x$ and $a \in y$. Hence $a$ is contained in every element of $\{x, y\}$. Therefore $a \in \bigcap \{x, y\}$. End. $\square$

FOUNDATIONS_10_7239895674257408

**Corollary 0.16.** Let $x$ be a set. Then $\bigcap \{x\} = x$.