

Curry's Paradox

Naproche formalization:

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Curry's Paradox is a paradox described by Haskell Curry in 1942 [Curry1942]. It allows the derivation of an arbitrary statement from a self-referential expression that presupposes its own validity.

Untyped λ -Calculus

Our formalization of Curry's Paradox in Naproche is based on an untyped λ -calculus:

Signature. A expression is a notion.

Let A, B denote expressions.

Signature. A variable is an expression.

Let n denote a natural number.

Signature. v_n is a variable.

Signature (Abstraction). $\lambda v_n. A$ is an expression.

Signature (Application). $(A)(B)$ is an expression.

Signature (Fixed-point combinator). `fix` is an expression such that

$$(\text{fix})(A) = (A)((\text{fix})(A))$$

for every expression A .

Propositional Logic

We extend this λ -calculus by a fragment of propositional logic:

Signature (Implication). $A \rightarrow B$ is an expression.

Signature (Truth). A is true is a relation.

Axiom (β -reduction). $(\lambda v_n. v_n \rightarrow B)(A) = A \rightarrow B$.

Axiom (Reflexivity). $A \rightarrow A$ is true.

Axiom (Modus Ponens). If A is true and $A \rightarrow B$ is true then B is true.

Axiom (Strengthening). If $A \rightarrow (A \rightarrow B)$ is true then $A \rightarrow B$ is true.

Curry's paradox

Using the fixpoint combinator from above we can formulate a self-referential expression X of the form “If X is true then A is true” for any arbitrary expression A by defining $X = (\mathbf{fix})(\lambda v_0. v_0 \rightarrow A)$. From the existence of such an expression X together with the above axioms we can then derive that any expression A is true.

Theorem (Curry's paradox). Every expression is true.

Proof. Let A be an expression. Take $N = \lambda v_0. v_0 \rightarrow A$ and $X = (\mathbf{fix})(N)$.

(1) Then $X = (N)(X) = X \rightarrow A$ (by β -reduction).

Hence $X \rightarrow (X \rightarrow A)$ is true (by 1, Reflexivity).

(2) Thus $X \rightarrow A$ is true (by Strengthening).

(3) Therefore X is true (by 1, 2).

Consequently A is true (by Modus Ponens, 2, 3). □

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