Regularity of successor cardinals

$\mathbb{N}$aproche formalization:

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This is a formalization of a theorem of Felix Hausdorff stating that successor cardinals are always regular.

In lack of a formalization in $\mathbb{N}$aproche of a proof of the fact that $|X \times X| = |X|$ for every set X which is required by the subsequent proof of Hausdorff's Theorem, we add an axiom that ensures that this fact holds in our theory:

Axiom. $|X \times X| = |X|$ for every set X.

The following result appears in [**Hausdorff1908**], where Hausdorff mentions that the proof of the regularity of successor cardinals is *"ganz einfach"* (*"very simple"*) and can be skipped.

Theorem (Hausdorff). Let κ be a cardinal. Then κ^+ is regular.
Proof by contradiction. Assume the contrary. Take a cofinal subset x of κ^+ such that $|x| \neq \kappa^+$. Then $|x| \leq \kappa$. Take a surjective map f from κ onto x (by `SET_THEORY_06_192336220913664`). Indeed x and κ are nonempty and $|\kappa| = \kappa$. Then $f(\xi) \in \kappa^+$ for all $\xi \in \kappa$.

Let us show that for all $z \in \kappa^+$ if z is nonempty then there exists a surjective map from κ onto z. Let $z \in \kappa^+$. Assume that z is nonempty. κ is nonempty. Hence the thesis (by `SET_THEORY_06_192336220913664`). Indeed $|\kappa| \geq |z|$. End.

Define

$$g(z) = \begin{cases} \text{choose } h : \kappa \twoheadrightarrow z \text{ in } h & : z \text{ has an element} \\ \text{const}^0_\kappa & : z \text{ has no element} \end{cases}$$

for z in κ^+.

Let us show that for all $\xi, \zeta \in \kappa$ $g(f(\xi))$ is a map such that $\zeta \in \mathrm{dom}(g(f(\xi)))$. Let $\xi, \zeta \in \kappa$. If $f(\xi)$ has an element then $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. If $f(\xi)$ has no element then $g(f(\xi)) = \text{const}^0_\kappa$. Hence $\mathrm{dom}(g(f(\xi))) = \kappa$. Therefore $\zeta \in \mathrm{dom}(g(f(\xi)))$. End.

For all objects ξ, ζ we have $\xi, \zeta \in \kappa$ iff $(\xi, \zeta) \in \kappa \times \kappa$. Define $h(\xi, \zeta) =$

$g(f(\xi))(\zeta)$ for $(\xi, \zeta) \in \kappa \times \kappa$.

Let us show that h is surjective onto κ^+.

Every element of κ^+ is an element of $h[\kappa \times \kappa]$.
Proof. Let n be an element of κ^+. Take an element ξ of κ such that $n < f(\xi)$. Take an element ζ of κ such that $g(f(\xi))(\zeta) = n$. Indeed $g(f(\xi))$ is a surjective map from κ onto $f(\xi)$. Then $n = h(\xi, \zeta)$. End.

Every element of $h[\kappa \times \kappa]$ is an element of κ^+.
Proof. Let n be an element of $h[\kappa \times \kappa]$. We can take elements a, b of κ such that $n = h(a, b)$. Then $n = g(f(a))(b)$. Indeed $(a, b) \in \kappa \times \kappa$ and $h(a, b) = g(f(a))(b)$. $f(a)$ is an element of κ^+. Every element of $f(a)$ is an element of κ^+.

Case $f(a)$ has an element. Then $g(f(a))$ is a surjective map from κ onto $f(a)$. Hence $n \in f(a) \in \kappa^+$. Thus $n \in \kappa^+$. End.

Case $f(a)$ has no element. Then $g(f(a)) = \mathrm{const}^0_\kappa$. Hence n is the empty set. Thus $n \in \kappa^+$. End. End.

Hence $\mathrm{range}(h) = h[\kappa \times \kappa] = \kappa^+$. End.

Therefore $|\kappa^+| \leq |\kappa \times \kappa|$ (by `SET_THEORY_06_192336220913664`). Indeed $\kappa \times \kappa$ and κ^+ are nonempty sets and h is a surjective map from $\kappa \times \kappa$ to κ^+. Consequently $\kappa^+ \leq \kappa$. Contradiction. $\square$