

Finite and Infinite Classes

Finite Classes

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Definition 0.1. Let X be a class and n be a natural number. X has n elements iff X is equinumerous to $\{1, \dots, n\}$.

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Definition 0.2. Let X be a class. X is finite iff there exists a natural number n such that X has n elements.

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Proposition 0.3. Let X, Y be classes. If X is finite and Y is equinumerous to X then Y is finite.

Proof. Assume that X is finite and Y is equinumerous to X . Take a natural number n and a bijection f between $\{1, \dots, n\}$ and X and a bijection g between X and Y . Then $g \circ f$ is a bijection between $\{1, \dots, n\}$ and Y (by FOUNDATIONS_08_6435206693126144). Indeed X, Y are classes. Hence Y is finite. \square

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Proposition 0.4. Let X be a class. X has 0 elements iff $X = \emptyset$.

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Proposition 0.5. Let X be a class. X has 1 element iff $X = \{a\}$ for some object a .

Proof. Case X has 1 element. Take a bijection f between $\{1, \dots, 1\}$ and X . We have $\{1, \dots, 1\} = \{1\}$. Hence $X = \{f(1)\}$. End.

Case $X = \{a\}$ for some object a . Consider an object a such that $X = \{a\}$. Define $f(x) = 1$ for $x \in \{a\}$. Then f is a bijection between $\{a\}$ and $\{1, \dots, 1\}$. End. \square

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Proposition 0.6. Let X be a class. X has 2 elements iff $X = \{a, b\}$ for some distinct objects a, b .

Proof. Case X has 2 elements. Take a bijection f between $\{1, \dots, 2\}$ and X . We have $\{1, \dots, 2\} = \{1, 2\}$. Hence $X = \{f(1), f(2)\}$. We have $f(1) \neq f(2)$. End.

Case $X = \{a, b\}$ for some distinct objects a, b . Consider distinct objects a, b such that $X = \{a, b\}$. Define

$$f(x) = \begin{cases} 1 & : x = a \\ 2 & : x = b \end{cases}$$

for $x \in \{a, b\}$. Then f is a bijection between $\{a, b\}$ and $\{1, \dots, 2\}$. Indeed f is injective and surjective onto $\{1, \dots, 2\}$. End. \square

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Proposition 0.7. Let n be a natural number and X be a class that has n elements and a be an object such that $a \notin X$. Then $X \cup \{a\}$ has $n + 1$ elements.

Proof. Take a bijection f between X and $\{1, \dots, n\}$. Define

$$g(x) = \begin{cases} f(x) & : x \in X \\ n + 1 & : x = a \end{cases}$$

for $x \in X \cup \{a\}$.

(1) g is a map from $X \cup \{a\}$ to $\{1, \dots, n + 1\}$.

Indeed we can show that $g(x) \in \{1, \dots, n + 1\}$ for all $x \in X \cup \{a\}$. Let $x \in X \cup \{a\}$. If $x \in X$ then $g(x) \in \{1, \dots, n\}$. If $x = a$ then $g(x) = n + 1$. End.

(2) g is injective.

(3) g is surjective onto $\{1, \dots, n + 1\}$.

Proof. Let $k \in \{1, \dots, n + 1\}$.

Case $k \leq n$. Then $k \in \{1, \dots, n\}$. Hence we can take a $x \in X$ such that $k = f(x)$. End.

Case $k = n + 1$. Then $k = g(a)$. End. Qed.

Hence g is a bijection between $X \cup \{a\}$ and $\{1, \dots, n + 1\}$. \square

Infinite Classes

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Definition 0.8. Let X be a class. X is infinite iff X is not finite.

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Proposition 0.9. Let X, Y be classes. If X is infinite and Y is equinumerous to X then Y is infinite.

Proof. Assume that Y is equinumerous to X . If Y is finite then X is finite. Hence if X is infinite then Y is infinite. \square