ω is a Limit Ordinal

SET_THEORY_03_764451995254784

Proposition 0.1. ω is a limit ordinal.

Proof. ω is transitive.
Proof. Define $\Phi = \{n \in \omega \mid \text{for all } m \in n \text{ we have } m \in \omega\}$.

(1) $0 \in \Phi$.

(2) For all $n \in \Phi$ we have $\mathrm{succ}(n) \in \Phi$.
Proof. Let $n \in \Phi$. Then every element of n is contained in ω. Hence every element of $\mathrm{succ}(n)$ is contained in ω. Thus $\mathrm{succ}(n) \in \Phi$. Qed.

Therefore $\omega \subseteq \Phi$. Consequently ω is transitive. Qed.

Every element of ω is an ordinal. Hence every element of ω is transitive. Thus ω is an ordinal.

ω is a limit ordinal.
Proof. Assume the contrary. We have $\omega \neq 0$. Hence ω is a successor ordinal. Take an ordinal α such that $\mathrm{succ}(\alpha) = \omega$. Then $\alpha \in \omega$. Thus $\omega = \mathrm{succ}(\alpha) \in \omega$. Contradiction. Qed. $\square$