

Invertible Maps

Definitions

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Definition 0.1. Let f be a map. An inverse of f is a map g from $\text{range}(f)$ to $\text{dom}(f)$ such that $f(a) = b$ iff $g(b) = a$ for all $a \in \text{dom}(f)$ and all $b \in \text{dom}(g)$.

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Definition 0.2. Let f be a map. f is invertible iff f has an inverse.

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Lemma 0.3. Let f be a map and g, g' be inverses of f . Then $g = g'$.

Proof. We have $\text{dom}(g) = \text{range}(f) = \text{dom}(g')$.

Let us show that $g(b) = g'(b)$ for all $b \in \text{range}(f)$. Let $b \in \text{range}(f)$. Take $a = g'(b)$. Then $g(b) = a$ iff $f(a) = b$. We have $f(a) = b$ iff $g'(b) = a$. Thus $g(b) = g'(b)$. End. \square

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Definition 0.4. Let f be an invertible map. f^{-1} is the inverse of f .

Let f is involutory stand for f is the inverse of f . Let f is selfinverse stand for f is the inverse of f .

Basic Properties

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Proposition 0.5. Let A, B be classes and $f : A \rightarrow B$ and $g : B \rightarrow A$. Then g is the inverse of f iff $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

Proof. Case g is the inverse of f . We have $\text{dom}(g \circ f) = \text{dom}(f) = A = \text{dom}(\text{id}_A)$. For all $a \in A$ we have $(g \circ f)(a) = g(f(a)) = a$. Hence $g \circ f = \text{id}_A$.

We have $\text{dom}(f \circ g) = \text{dom}(g) = B = \text{dom}(\text{id}_B)$. For all $b \in B$ we have $(f \circ g)(b) = f(g(b)) = b$. Hence $f \circ g = \text{id}_B$. End.

Case $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$. Then $\text{dom}(g) = B = \text{range}(f)$ and

$\text{range}(g) = A = \text{dom}(f)$. Let $a \in \text{dom}(f)$ and $b \in \text{dom}(g)$. If $f(a) = b$ then $g(b) = g(f(a)) = (g \circ f)(a) = \text{id}_A(a) = a$. If $g(b) = a$ then $f(a) = f(g(b)) = (f \circ g)(b) = \text{id}_B(b) = b$. Hence $f(a) = b$ iff $g(b) = a$. End. \square

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Proposition 0.6. Let A, B be classes and $f : A \rightarrow B$. Assume that f is invertible. Then f^{-1} is an invertible surjective map from B onto A such that $(f^{-1})^{-1} = f$.

Proof. f^{-1} is a map from B to A . Indeed $\text{range}(f) = B$ and $\text{dom}(f) = A$. f^{-1} is surjective onto A . Indeed for any $a \in A$ we have $f^{-1}(f(a)) = a$. f^{-1} is the inverse of f . Thus $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$. Therefore f is the inverse of f^{-1} . \square

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Proposition 0.7. Let A, B be classes and $f : A \rightarrow B$. Assume that f is invertible. Then $f \circ f^{-1} = \text{id}_B$ and $f^{-1} \circ f = \text{id}_A$.

Proof. f^{-1} is a surjective map from B onto A . f^{-1} is the inverse of f . \square

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Proposition 0.8. Let A, B be classes and $f : A \rightarrow B$ and $a \in A$. Assume that f is invertible. Then $f^{-1}(f(a)) = a$.

Proof. We have $f^{-1}(f(a)) = (f^{-1} \circ f)(a) = \text{id}_A(a) = a$. \square

Proposition 0.9. Let A, B be classes and $f : A \rightarrow B$ and $b \in B$. Assume that f is invertible. Then $f(f^{-1}(b)) = b$.

Proof. We have $f(f^{-1}(b)) = (f \circ f^{-1})(b) = \text{id}_B(b) = b$. \square

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Proposition 0.10. Let A, B, C be classes and $f : A \rightarrow B$ and $g : B \rightarrow C$. Assume that f and g are invertible. Then $g \circ f$ is invertible and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. f^{-1} is a surjective map from B onto A . g^{-1} is a surjective map from C onto B . Take $h = f^{-1} \circ g^{-1}$. Then h is a surjective map from C

onto A (by FOUNDATIONS_08_8542698338254848). $g \circ f$ is a map from A to C .

Let us show that $((g \circ f) \circ h) = \text{id}_C$. We have $f \circ (f^{-1} \circ g^{-1}) = (f \circ f^{-1}) \circ g^{-1}$. Indeed $f \circ (f^{-1} \circ g^{-1})$ and $(f \circ f^{-1}) \circ g^{-1}$ are maps of C . $f \circ h$ is a map from C to B . Hence

$$\begin{aligned} & (g \circ f) \circ h \\ &= g \circ (f \circ h) \\ &= g \circ (f \circ (f^{-1} \circ g^{-1})) \\ &= g \circ ((f \circ f^{-1}) \circ g^{-1}) \\ &= g \circ (\text{id}_B \circ g^{-1}) \\ &= g \circ g^{-1} \\ &= \text{id}_C. \end{aligned}$$

End.

Let us show that $h \circ (g \circ f) = \text{id}_A$. We have $(f^{-1} \circ g^{-1}) \circ g = f^{-1} \circ (g^{-1} \circ g)$. $g \circ f$ is a map from A to C . Hence

$$\begin{aligned} & h \circ (g \circ f) \\ &= (h \circ g) \circ f \\ &= ((f^{-1} \circ g^{-1}) \circ g) \circ f \\ &= (f^{-1} \circ (g^{-1} \circ g)) \circ f \\ &= (f^{-1} \circ \text{id}_B) \circ f \\ &= f^{-1} \circ f \\ &= \text{id}_A. \end{aligned}$$

End.

Thus h is the inverse of $g \circ f$. Indeed $g \circ f$ is a surjective map from A onto C and h is a surjective map from C onto A . \square

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Proposition 0.11. Let A, B be classes and $f : A \rightarrow B$ and $X \subseteq A$. Assume that f is invertible. Then $f \upharpoonright X$ is invertible and $(f \upharpoonright X)^{-1} = f^{-1} \upharpoonright (f_*(X))$.

Proof. $f \upharpoonright X$ is a surjective map from X onto $f_*(X)$. Take $g = f^{-1} \upharpoonright (f_*(X))$. Then g is a map of $f_*(X)$.

Let us show that $X \subseteq \text{range}(g)$. Let $a \in X$. Then $f(a) \in f_*(X)$. Hence

$g(f(a)) = f^{-1}(f(a)) = a$. Thus a is a value of g . End.

Let us show that $\text{range}(g) \subseteq X$. Let $a \in \text{range}(g)$. Take $b \in f_*(X)$ such that $a = g(b)$. Take $c \in X$ such that $b = f(c)$. Then $a = (f^{-1} \upharpoonright (f_*(X)))(b) = f^{-1}(b) = f^{-1}(f(c)) = c$. Hence $a \in X$. End.

Hence $\text{range}(g) = X$. Thus g is a surjective map onto X .

Let us show that $g((f \upharpoonright X)(a)) = a$ for all $a \in X$. Let $a \in X$. Then $g((f \upharpoonright X)(a)) = g(f(a)) = (f^{-1} \upharpoonright (f_*(X)))(f(a)) = f^{-1}(f(a)) = a$. End.

Let us show that $((f \upharpoonright X)(g(b))) = b$ for all $b \in f_*(X)$. Let $b \in f_*(X)$. Take $a \in X$ such that $b = f(a)$. We have $g(b) = g(f(a)) = (f^{-1} \upharpoonright (f_*(X)))(f(a)) = f^{-1}(f(a)) = a$. Hence $(f \upharpoonright X)(g(b)) = (f \upharpoonright X)(a) = f(a) = b$. End.

Thus $g \circ (f \upharpoonright X) = \text{id}_X$ and $(f \upharpoonright X) \circ g = \text{id}_{f_*(X)}$. Therefore g is the inverse of $f \upharpoonright X$. \square

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Proposition 0.12. Let A, B be classes and $f : A \rightarrow B$ and $Y \subseteq B$. Assume that f is invertible. Then

$$f^*(Y) = (f^{-1})_*(Y).$$

Proof. We have $(f^{-1})_*(Y) = \{f^{-1}(b) \mid b \in Y\}$ and $f^*(Y) = \{a \in A \mid f(a) \in Y\}$.

Let us show that $f^*(Y) \subseteq (f^{-1})_*(Y)$. Let $a \in f^*(Y)$. Take $b \in Y$ such that $b = f(a)$. Then $f^{-1}(b) = f^{-1}(f(a)) = a$. Hence $a \in (f^{-1})_*(Y)$. End.

Let us show that $(f^{-1})_*(Y) \subseteq f^*(Y)$. Let $a \in (f^{-1})_*(Y)$. Take $b \in Y$ such that $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = b$. Hence $a \in f^*(Y)$. End. \square

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Corollary 0.13. Let A, B be classes and $f : A \rightarrow B$ and $b \in B$. Assume that f is invertible. Then

$$f^*({b}) = \{f^{-1}(b)\}.$$

Proof. $f^*({b}) = f_*^{-1}({b})$. We have $f_*^{-1}({b}) = \{f^{-1}(c) \mid c \in {b}\}$. Hence $f_*^{-1}({b}) = \{f^{-1}(b)\}$. \square

Proposition 0.14. Let A, B be classes and $f : A \rightarrow B$. Then f is invertible iff f is injective.

Proof. Case f is invertible. Let $a, b \in A$. Assume $f(a) = f(b)$. Then $a = f^{-1}(f(a)) = f^{-1}(f(b)) = b$. End.

Case f is injective. Define $g(b) =$ “choose $a \in A$ such that $f(a) = b$ in A ” for $b \in B$. Then g is a map from B to A . For all $a \in A$ we have $a = g(f(a))$. Hence g is a surjective map from B onto A . For all $a \in A$ we have $g(f(a)) = a$. For all $b \in B$ we have $f(g(b)) = b$. Hence g is the inverse of f . End. \square

Corollary 0.15. Let A, B be classes and $f : A \rightarrow B$. Assume that f is invertible. Then f^{-1} is a bijection between B and A .

Proof. f^{-1} is a surjective map from B onto A . f^{-1} is invertible. Hence f^{-1} is injective. Therefore f^{-1} is a bijection between B and A . \square

Involutions

Definition 0.16. Let A be a class. An involution on A is a selfinverse map f on A .

Proposition 0.17. Let A be a class. id_A is an involution on A .

Proof. We have $\text{id}_A \circ \text{id}_A = \text{id}_A$. Hence id_A is selfinverse. \square

Proposition 0.18. Let A be a class and f, g be involutions on A . Then $g \circ f$ is an involution on A iff $g \circ f = f \circ g$.

Proof. Case $g \circ f$ is an involution on A . Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1} = f \circ g$. End.

Case $g \circ f = f \circ g$. $f \circ f$, $f \circ g$ and $f \circ g$ are maps on A . Hence

$$(g \circ f) \circ (g \circ f)$$

$$\begin{aligned}
&= (g \circ f) \circ (f \circ g) \\
&= ((g \circ f) \circ f) \circ g \\
&= (g \circ (f \circ f)) \circ g \\
&= (g \circ \text{id}_A) \circ g \\
&= g \circ g \\
&= \text{id}_A.
\end{aligned}$$

Thus $g \circ f$ is selfinverse. End. □

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Corollary 0.19. Let A be a class and f be an involutions on A . Then $f \circ f$ is an involution on A .

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Proposition 0.20. Let A be a class and f be an involution on A . Then f is a permutation of A .

Proof. f is an invertible map of A that surjects onto A . Hence f is a bijection between A and A . Thus f is a permutation of A . □