Transfinite Recursion Theorem

ℕaproche formalization:

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2024

This is a formalization of the *Transfinite Recursion Theorem* (cf. [**Koepke2018**]). It states that for any map $G : A^{<\infty} \to A$, where $A^{<\infty}$ denotes the class of all maps $\alpha \to A$ for some ordinal α, there exists a unique map $F : \mathbf{Ord} \to A$ that is *recursive regarding* G, i.e.

$$F(\alpha) = G(F \upharpoonright \alpha)$$

for all ordinals α.

Lemma (Coincidence Lemma). Let A be a class and G be a map from $A^{<\infty}$ to A. Let H, H' be maps to A that are recursive regarding G. Then

$$H(\alpha) = H'(\alpha)$$

for all $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$.

Proof. Define $\Phi = \{\alpha \in \mathbf{Ord} \mid \text{if } \alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H') \text{ then } H(\alpha) = H'(\alpha)\}$.

For all ordinals α if every ordinal less than α lies in Φ then $\alpha \in \Phi$.
Proof. Let $\alpha \in \mathbf{Ord}$. Assume that every $y \in \alpha$ lies in Φ.

Let us show that if $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$ then $H(\alpha) = H'(\alpha)$. Suppose $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Then $\alpha \subseteq \mathrm{dom}(H), \mathrm{dom}(H')$. Indeed $\mathrm{dom}(H)$ and $\mathrm{dom}(H')$ are transitive classes. Hence for all $y \in \alpha$ we have $y \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Thus $H(y) = H'(y)$ for all $y \in \alpha$. Therefore $H \upharpoonright \alpha = H' \upharpoonright \alpha$. H and H' are recursive regarding G. We have $H \upharpoonright \alpha, H' \upharpoonright \alpha \in A^{<\infty}$. Hence $H(\alpha) = G(H \upharpoonright \alpha) = G(H' \upharpoonright \alpha) = H'(\alpha)$. End.

Thus $\alpha \in \Phi$. Qed.

[prover vampire] Then Φ contains every ordinal (by `SET_THEORY_02_8493935460614144`). Therefore we have $H(\alpha) = H'(\alpha)$ for all $\alpha \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. ∎

Theorem (Transfinite Recursion Theorem: Existence). Let A be a class and G be a map from $A^{<\infty}$ to A. Then there exists a map F from **Ord** to A that is recursive regarding G.
Proof. Every ordinal is contained in the domain of some map H to A such that H is recursive regarding G.
Proof. Define

$$\Phi = \left\{ \alpha \in \mathbf{Ord} \;\middle|\; \begin{array}{l} \alpha \text{ is contained in the domain of some map to } A \text{ that is} \\ \text{recursive regarding } G \end{array} \right\}.$$

Let us show that for every ordinal α if every ordinal less than α lies in Φ then $\alpha \in \Phi$. Let α be an ordinal. Assume that every ordinal less than α lies in Φ. Then for all $y \in \alpha$ there exists a map h to A such that h is recursive regarding G and $y \in \mathrm{dom}(h)$. Define $H'(y) =$ "choose a map h to A such that h is recursive regarding G and $y \in \mathrm{dom}(h)$ in $h(y)$" for $y \in \alpha$. Then H' is a map from α to A. We have $H' = H' \restriction \alpha$. Define

$$H(\beta) = \begin{cases} H'(\beta) & : \beta < \alpha \\ G(H' \restriction \beta) & : \beta = \alpha \end{cases}$$

for $\beta \in \mathrm{succ}(\alpha)$.

Let us show that $H \restriction \beta \in A^{<\infty}$ for all $\beta \in \mathrm{dom}(H)$. Let $\beta \in \mathrm{dom}(H)$. $\mathrm{dom}(H \restriction \beta) = \beta$ and $(H \restriction \beta)(b) = H(b)$ for all $b \in \beta$. $H(b) \in A$ for all $b \in \beta$. Hence $H \restriction \beta$ is a map from β to A. End.

(a) $\mathrm{dom}(H)$ is a transitive subclass of **Ord**.

(b) For all $\beta \in \mathrm{dom}(H)$ we have $H(\beta) = G(H \restriction \beta)$.
Proof. Let $\beta \in \mathrm{dom}(H)$. Then $\beta < \alpha$ or $\beta = \alpha$.

Case $\beta < \alpha$. Choose a map h to A such that h is recursive regarding G and $\beta \in \mathrm{dom}(h)$ and $H'(\beta) = h(\beta)$.

Let us show that for all $y \in \beta$ we have $h(y) = H(y)$. Let $y \in \beta$. Then $H(y) = H'(y)$. Choose a map h' to A such that h' is recursive regarding G and $y \in \mathrm{dom}(h')$ and $H'(y) = h'(y)$. [prover vampire] Then $h'(y) = h(y)$ (by Coincidence Lemma). Indeed $y \in \mathrm{dom}(h) \cap \mathrm{dom}(h')$. End.

Hence $h \restriction \beta = H \restriction \beta$. Thus $H(\beta) = H'(\beta) = h(\beta) = G(h \restriction \beta) = G(H \restriction \beta)$. End.

Case $\beta = \alpha$. We have $H \restriction \alpha = H' \restriction \alpha$. End. Qed.

Hence H is a map to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$. Thus $\alpha \in \Phi$. End.

[prover vampire] Therefore Φ contains every ordinal (by `SET_THEORY_02_8493935460614144`). Consequently every ordinal is contained in the domain of some map H to A such that H is recursive regarding G. Qed.

Define $F(\alpha) =$ "choose a map H to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$ in $H(\alpha)$" for $\alpha \in \mathbf{Ord}$. Then F is a map from $\mathbf{Ord}$ to A.

F is recursive regarding G.
Proof. (a) $\mathrm{dom}(F)$ is a transitive subclass of $\mathbf{Ord}$.

(b) For all $\alpha \in \mathbf{Ord}$ we have $F(\alpha) = G(F \restriction \alpha)$.
Proof. Let $\alpha \in \mathbf{Ord}$. Choose a map H to A such that H is recursive regarding G and $\alpha \in \mathrm{dom}(H)$ and $F(\alpha) = H(\alpha)$.

Let us show that $F(\beta) = H(\beta)$ for all $\beta \in \alpha$. Let $\beta \in \alpha$. Choose a map H' to A such that H' is recursive regarding G and $\beta \in \mathrm{dom}(H')$ and $F(\beta) = H'(\beta)$. [prover vampire] Then $H(\beta) = H'(\beta)$ (by Coincidence Lemma). Indeed $\beta \in \mathrm{dom}(H) \cap \mathrm{dom}(H')$. Therefore $F(\beta) = H'(\beta)$. End.

Hence $H \restriction \alpha = F \restriction \alpha$. Thus $F(\alpha) = H(\alpha) = G(H \restriction \alpha) = G(F \restriction \alpha)$. Qed.
Qed. □

Theorem (Transfinite Recursion Theorem: Uniqueness). Let A be a class and G be a map from $A^{<\infty}$ to A. Let F, F' be maps from $\mathbf{Ord}$ to A that are recursive regarding G. Then $F = F'$.
Proof prover vampire. F and F' are recursive regarding G. Then $F(\alpha) = F'(\alpha)$ for all $\alpha \in \mathrm{dom}(F) \cap \mathrm{dom}(F')$ (by Coincidence Lemma). We have $\mathrm{dom}(F) = \mathbf{Ord} = \mathrm{dom}(F')$. Hence $F(\alpha) = F'(\alpha)$ for all $\alpha \in \mathbf{Ord}$. Thus $F = F'$. □

As a corollary of the transfinite recursion theorem we get that we can define maps recursively on the ordinals by case distinction: For given maps $G : \mathbf{Ord} \times A \to A$ and $H : \mathbf{Ord} \times A^{<\infty} \to A$ and an element $a \in A$ we can define a map $F : \mathbf{Ord} \to A$ by

- $F(0) = a$,
- $F(\mathrm{succ}(\alpha)) = G(\alpha, F(\alpha))$, and
- $F(\lambda) = H(\lambda, F \restriction \lambda)$ for any limit ordinal λ.

To establish the well-formedness of the conclusion of that corollary in Naproche, we need two additional lemmas:

Lemma. Let A be a class and α be an ordinal and $F : \mathbf{Ord} \to A$. Then $(\alpha, F(\alpha)) \in \mathbf{Ord} \times A$.

Lemma. Let A be a class and λ be a limit ordinal and $F : \mathbf{Ord} \to A$. Then $(\lambda, F \restriction \lambda) \in \mathbf{Ord} \times A^{<\infty}$.

Corollary. Let A be a class. Let $a \in A$ and $G : \mathbf{Ord} \times A \to A$ and $H : \mathbf{Ord} \times A^{<\infty} \to A$. Then there exists a map F from **Ord** to A such that

$$F(0) = a$$

and for all ordinals α we have

$$F(\mathrm{succ}(\alpha)) = G(\alpha, F(\alpha))$$

and for all limit ordinals λ we have

$$F(\lambda) = H(\lambda, F \restriction \lambda).$$

Proof. $(\mathrm{pred}(\mathrm{dom}(f)), f(\mathrm{pred}(\mathrm{dom}(f)))) \in \mathbf{Ord} \times A$ for all $f \in A^{<\infty}$ such that $\mathrm{dom}(f)$ is a successor ordinal.

Define

$$J(f) = \begin{cases} a & : \mathrm{dom}(f) = 0 \\ G(\mathrm{pred}(\mathrm{dom}(f)), f(\mathrm{pred}(\mathrm{dom}(f)))) & : \mathrm{dom}(f) \text{ is a successor ordinal} \\ H(\mathrm{dom}(f), f) & : \mathrm{dom}(f) \text{ is a limit ordinal} \end{cases}$$

for $f \in A^{<\infty}$.

Then J is a map from $A^{<\infty}$ to A. Indeed we can show that for any $f \in A^{<\infty}$ we have $J(f) \in A$. Let $f \in A^{<\infty}$. Take $\alpha \in \mathbf{Ord}$ such that $f : \alpha \to A$. If $\alpha = 0$ then $J(f) = a \in A$. If α is a successor ordinal then $J(f) = G(\mathrm{pred}(\alpha), f(\mathrm{pred}(\alpha))) \in A$. [prover vampire] If α is a limit ordinal then $J(f) = H(\alpha, f) \in A$. End.

Hence we can take a map F from **Ord** to A that is recursive regarding J. Then $F \restriction \alpha \in A^{<\infty}$ for any ordinal α.

(1) $F(0) = a$.
Proof. $F(0) = J(F \restriction 0) = a$. Qed.

(2) $F(\mathrm{succ}(\alpha)) = G(\alpha, F(\alpha))$ for all ordinals α.
Proof. Let α be an ordinal. Then $F(\mathrm{succ}(\alpha)) = J(F \restriction \mathrm{succ}(\alpha)) = G(\mathrm{pred}(\mathrm{succ}(\alpha)), (F \restriction \mathrm{succ}(\alpha))(\mathrm{pred}(\mathrm{succ}(\alpha)))) = G(\alpha, (F \restriction \mathrm{succ}(\alpha))(\alpha)) = G(\alpha, F(\alpha))$. Qed.

(3) $F(\lambda) = H(\lambda, F \restriction \lambda)$ for all limit ordinals λ.
Proof. Let λ be a limit ordinal. Then $F(\lambda) = J(F \restriction \lambda) = H(\lambda, F \restriction \lambda)$. Qed.

Hence the thesis (by 1, 2, 3). $\square$

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