Natural Numbers

SET_THEORY_03_4310076227584000

Definition 0.1.

$$\omega = \left\{ n \in \mathbf{Ord} \;\middle|\; \begin{array}{l} n \in X \text{ for every } X \subseteq \mathbf{Ord} \text{ such that } 0 \in X \text{ and} \\ \text{for all } x \in X \text{ we have } \mathrm{succ}(x) \in X \end{array} \right\}.$$

Let a natural number stand for an element of ω.

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Proposition 0.2. $0 \in \omega$.

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Proposition 0.3. Let $n \in \omega$. Then $\mathrm{succ}(n) \in \omega$.

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Proposition 0.4. Let $\Phi \subseteq \omega$. Assume that $0 \in \Phi$ and for every $x \in \Phi$ we have $\mathrm{succ}(x) \in \Phi$. Then $\Phi = \omega$.

Proof. Suppose $\Phi \neq \omega$. Consider an element n of ω that is not contained in Φ. Take $\Phi' = \Phi \setminus \{n\}$.

(1) $0 \in \Phi'$. Indeed $0 \in \Phi$ and $0 \neq n$.

(2) For each $x \in \Phi'$ we have $\mathrm{succ}(x) \in \Phi'$.
Proof. Let $x \in \Phi'$. Then $\mathrm{succ}(x) \in \Phi$.

Let us show that $\mathrm{succ}(x) \neq n$. Assume $\mathrm{succ}(x) = n$. Then $x \notin \Phi$. Indeed $n \notin \Phi$ and if $x \in \Phi$ then $n = \mathrm{succ}(x) \in \Phi$. Contradiction. End.

Thus $\mathrm{succ}(x) \in \Phi'$. Qed.

Therefore every element of ω lies in Φ'. Indeed $\Phi' \subseteq \mathbf{Ord}$. Consequently $n \in \Phi'$. Contradiction. □

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Corollary 0.5. ω is a set.

Proof. Define $f(n) = \mathrm{succ}(n)$ for $n \in \omega$. Take a subset X of ω that is inductive regarding 0 and f. Indeed f is a map from ω to ω. Then we have $0 \in X$ and for each $n \in X$ we have $\mathrm{succ}(n) \in X$. Thus $X = \omega$. Therefore

ω is a set. □

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Proposition 0.6. Let $n \in \omega$. Then $n = 0$ or $n = \text{succ}(m)$ for some $m \in \omega$.

Proof. Assume the contrary. Consider a $k \in \omega$ such that neither $k = 0$ nor $k = \text{succ}(m)$ for some $m \in \omega$. Take a class ω' such that $\omega' = \omega \setminus \{k\}$. Then ω' is a set.

(1) $0 \in \omega'$. Indeed $k \neq 0$.

(2) For all $m \in \omega'$ we have $\text{succ}(m) \in \omega'$.
Proof. Let $m \in \omega'$. Then $\text{succ}(m) \neq k$. Hence $\text{succ}(m) \in \omega'$. Qed.

Thus every element of ω is contained in ω'. Therefore $k \in \omega'$. Contradiction. □

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Proposition 0.7. Every element of ω is an ordinal.

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Definition 0.8. $1 = \text{succ}(0)$.

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Definition 0.9. $2 = \text{succ}(1)$.

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Proposition 0.10. $1 = \{0\}$.

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Proposition 0.11. $2 = \{0, 1\}$.