

# Zermelo's Well-ordering Theorem

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This is a formalization of *Zermelo's Well-ordering Theorem*, i.e. of the assertion that under the assumption of the axiom of choice every set is equinumerous to some ordinal number, where an ordinal number is regarded as a transitive set whose elements are transitive sets as well. The proof of this theorem presented here is oriented on [Koepke2018].

**Axiom (Axiom of Choice).** Let  $X$  be a system of nonempty sets. Then there exists a choice map for  $X$ .

In the following, for any class  $A$ , we write  $A^{<\infty}$  to denote the collection of all maps  $f : \alpha \rightarrow A$  for some ordinal  $\alpha$ . Moreover, for any map  $G : A^{<\infty} \rightarrow A$  we say that a map  $F : \mathbf{Ord} \rightarrow A$ , where  $\mathbf{Ord}$  denotes the class of all ordinals, is recursive regarding  $G$  if  $F(\alpha) = G(F \upharpoonright \alpha)$  for all  $\alpha \in \mathbf{Ord}$ .

**Theorem (Zermelo's Well-Ordering Theorem).** Every set is equinumerous to some ordinal.

*Proof.* Let  $x$  be a set. Consider a choice map  $g$  for  $\mathcal{P}(x) \setminus \{\emptyset\}$ . Take  $A = x \cup \{x\}$ . Every  $F \in A^{<\infty}$  is a map from some ordinal to  $A$ . For any  $F \in A^{<\infty}$  if  $x \setminus \text{range}(F) \neq \emptyset$  then  $x \setminus \text{range}(F) \in \text{dom}(g)$ . Indeed  $x \setminus \text{range}(F)$  is a subset of  $x$  for any  $F \in A^{<\infty}$ . Define

$$G(F) = \begin{cases} g(x \setminus \text{range}(F)) & : x \setminus \text{range}(F) \neq \emptyset \\ x & : x \setminus \text{range}(F) = \emptyset \end{cases}$$

for  $F \in A^{<\infty}$ . We can show that for any  $F \in A^{<\infty}$  if  $x \setminus \text{range}(F) \neq \emptyset$  then  $G(F) \in x \setminus \text{range}(F)$ . Let  $F \in A^{<\infty}$ . Assume  $x \setminus \text{range}(F) \neq \emptyset$ . Then  $G(F) \in x \setminus \text{range}(F)$ . End.  $G$  is a map from  $A^{<\infty}$  to  $A$ . Indeed we can show that for any  $F \in A^{<\infty}$  we have  $G(F) \in A$ . Let  $F \in A^{<\infty}$ . If  $x \setminus \text{range}(F) \neq \emptyset$  then  $G(F) \in x \setminus \text{range}(F)$ . If  $x \setminus \text{range}(F) = \emptyset$  then  $G(F) = x$ . Hence  $G(F) \in A$ . End. Hence we can take a map  $F$  from  $\mathbf{Ord}$  to  $A$  that is recursive regarding  $G$ . For any ordinal  $\alpha$  we have  $F \upharpoonright \alpha \in A^{<\infty}$ .

For any  $\alpha \in \mathbf{Ord}$  we have

$$x \setminus F[\alpha] \neq \emptyset \implies F(\alpha) \in x \setminus F[\alpha]$$

and

$$x \setminus F[\alpha] = \emptyset \implies F(\alpha) = x.$$

Proof. Let  $\alpha \in \mathbf{Ord}$ . We have  $F[\alpha] = \{F(\beta) \mid \beta \in \alpha\}$ . Hence  $F[\alpha] = \{G(F \upharpoonright \beta) \mid \beta \in \alpha\}$ . We have  $\text{range}(F \upharpoonright \alpha) = \{F(\beta) \mid \beta \in \alpha\}$ . Thus  $\text{range}(F \upharpoonright \alpha) = F[\alpha]$ .

Case  $x \setminus F[\alpha] \neq \emptyset$ . Then  $x \setminus \text{range}(F \upharpoonright \alpha) \neq \emptyset$ . Hence  $F(\alpha) = G(F \upharpoonright \alpha) \in x \setminus \text{range}(F \upharpoonright \alpha) = x \setminus F[\alpha]$ . End.

Case  $x \setminus F[\alpha] = \emptyset$ . Then  $x \setminus \text{range}(F \upharpoonright \alpha) = \emptyset$ . Hence  $F(\alpha) = G(F \upharpoonright \alpha) = x$ . End. Qed.

(1) For any ordinals  $\alpha, \beta$  such that  $\alpha < \beta$  and  $F(\beta) \neq x$  we have  $F(\alpha), F(\beta) \in x$  and  $F(\alpha) \neq F(\beta)$ .

Proof. Let  $\alpha, \beta \in \mathbf{Ord}$ . Assume  $\alpha < \beta$  and  $F(\beta) \neq x$ . Then  $x \setminus F[\beta] \neq \emptyset$ .

(a) Hence  $F(\beta) \in x \setminus F[\beta]$ . We have  $F[\alpha] \subseteq F[\beta]$ . Thus  $x \setminus F[\alpha] \neq \emptyset$ . (b) Therefore  $F(\alpha) \in x \setminus F[\alpha]$ . Consequently  $F(\alpha), F(\beta) \in x$  (by a, b). We have  $F(\alpha) \in F[\beta]$  and  $F(\beta) \notin F[\beta]$ . Thus  $F(\alpha) \neq F(\beta)$ . Qed.

(2) There exists an ordinal  $\alpha$  such that  $F(\alpha) = x$ .

Proof. Assume the contrary. Then  $F$  is a map from  $\mathbf{Ord}$  to  $x$ .

Let us show that  $F$  is injective. Let  $\alpha, \beta \in \mathbf{Ord}$ . Assume  $\alpha \neq \beta$ . Then  $\alpha < \beta$  or  $\beta < \alpha$ . Hence  $F(\alpha) \neq F(\beta)$  (by 1). Indeed  $F(\alpha), F(\beta) \neq x$ . End.

Thus  $F$  is an injective map from some proper class to some set. Contradiction. Qed.

Define  $\Phi = \{\alpha \in \mathbf{Ord} \mid F(\alpha) = x\}$ .  $\Phi$  is nonempty. Hence we can take an  $\alpha \in \Phi$  such that  $\alpha \leq \beta$  for all  $\beta \in \Phi$ . Indeed  $\Phi$  has an element  $\alpha$  such that for no  $\beta \in \Phi$  we have  $\beta < \alpha$ . Take  $f = F \upharpoonright \alpha$ . Then  $f$  is a map from  $\alpha$  to  $x$ . Indeed for no  $\beta \in \alpha$  we have  $F(\beta) = x$ .

(3)  $f$  is surjective onto  $x$ .

Proof.  $x \setminus F[\alpha] = \emptyset$ . Hence  $\text{range}(f) = f[\alpha] = F[\alpha] = x$ . Qed.

(4)  $f$  is injective.

Proof. Let  $\beta, \gamma \in \alpha$ . Assume  $\beta \neq \gamma$ . We have  $f(\beta), f(\gamma) \neq x$ . Hence  $f(\beta) \neq f(\gamma)$  (by 1). Indeed  $\beta < \gamma$  or  $\gamma < \beta$ . Qed.

Therefore  $f$  is a bijection between  $\alpha$  and  $x$ . Consequently  $x$  and  $\alpha$  are equinumerous.  $\square$

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