

Isabelle/CTT — Constructive Type Theory with extensional equality and without universes

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```

theory CTT
imports Pure
begin

```

1 Constructive Type Theory: axiomatic basis

ML-file $\langle \sim \sim / \text{src} / \text{Provers} / \text{typedsimp} . \text{ML} \rangle$

setup *Pure-Thy.old-appl-syntax-setup*

```

typeddecl i
typeddecl t
typeddecl o

```

consts

— Judgments

```

Type    :: t  $\Rightarrow$  prop      ( $\langle \langle \text{notation} = \langle \text{postfix Type} \rangle - \text{type} \rangle \rangle [10] 5$ )
Eqtype  :: [t,t]  $\Rightarrow$  prop  ( $\langle \langle \text{notation} = \langle \text{infix Eqtype} \rangle - = / - \rangle \rangle [10,10] 5$ )
Elem    :: [i, t]  $\Rightarrow$  prop  ( $\langle \langle \text{notation} = \langle \text{infix Elem} \rangle - / : - \rangle \rangle [10,10] 5$ )
Egelem  :: [i,i,t]  $\Rightarrow$  prop ( $\langle \langle \text{notation} = \langle \text{mixfix Egelem} \rangle - = / - : / - \rangle \rangle [10,10,10]$ 

```

5)

```

Reduce  :: [i,i]  $\Rightarrow$  prop  ( $\langle \text{Reduce}[-,-] \rangle$ )

```

— Types for truth values

```

F       :: t
T       :: t      — F is empty, T contains one element
contr   :: i  $\Rightarrow$  i
tt      :: i

```

— Natural numbers

```

N       :: t
Zero    :: i      ( $\langle 0 \rangle$ )
succ    :: i  $\Rightarrow$  i
rec     :: [i, i, [i,i]  $\Rightarrow$  i]  $\Rightarrow$  i

```

— Binary sum

```

Plus    :: [t,t]  $\Rightarrow$  t      (infixr  $\langle + \rangle$  40)
inl     :: i  $\Rightarrow$  i
inr     :: i  $\Rightarrow$  i
when    :: [i, i  $\Rightarrow$  i, i  $\Rightarrow$  i]  $\Rightarrow$  i

```

— General sum and binary product

```

Sum     :: [t, i  $\Rightarrow$  t]  $\Rightarrow$  t
pair    :: [i,i]  $\Rightarrow$  i      ( $\langle \langle \text{indent} = 1 \text{ notation} = \langle \text{mixfix pair} \rangle \langle -, / - \rangle \rangle \rangle$ )
fst     :: i  $\Rightarrow$  i
snd     :: i  $\Rightarrow$  i
split   :: [i, [i,i]  $\Rightarrow$  i]  $\Rightarrow$  i

```

— General product and function space

```

Prod    :: [t, i  $\Rightarrow$  t]  $\Rightarrow$  t
lambda  :: (i  $\Rightarrow$  i)  $\Rightarrow$  i   (binder  $\langle \lambda \rangle$  10)
app     :: [i,i]  $\Rightarrow$  i      (infixl  $\langle ' \rangle$  60)

```

— Equality type
 $Eq \quad :: [t, i, i] \Rightarrow t$
 $eq \quad :: i$

Some inexplicable syntactic dependencies; in particular, "0" must be introduced after the judgment forms.

syntax

$-PROD \quad :: [idt, t, t] \Rightarrow t \quad (\langle (\langle indent=3 \text{ notation}=\langle binder \prod \rangle \prod \text{ } \vdash \cdot \cdot / \cdot \rangle \rangle 10)$
 $-SUM \quad :: [idt, t, t] \Rightarrow t \quad (\langle (\langle indent=3 \text{ notation}=\langle binder \sum \rangle \sum \text{ } \vdash \cdot \cdot / \cdot \rangle \rangle 10)$

syntax-consts

$-PROD \Leftrightarrow Prod$ **and**
 $-SUM \Leftrightarrow Sum$

translations

$\prod x:A. B \Leftrightarrow CONST Prod(A, \lambda x. B)$
 $\sum x:A. B \Leftrightarrow CONST Sum(A, \lambda x. B)$

abbreviation $Arrow \quad :: [t, t] \Rightarrow t \quad (\textbf{infixr} \langle \longrightarrow \rangle 30)$
where $A \longrightarrow B \equiv \prod \vdash A. B$

abbreviation $Times \quad :: [t, t] \Rightarrow t \quad (\textbf{infixr} \langle \times \rangle 50)$
where $A \times B \equiv \sum \vdash A. B$

Reduction: a weaker notion than equality; a hack for simplification. *Reduce* $[a, b]$ means either that $a = b : A$ for some A or else that a and b are textually identical.

Does not verify $a:A!$ Sound because only *trans-red* uses a *Reduce* premise. No new theorems can be proved about the standard judgments.

axiomatization

where

refl-red: $\bigwedge a. Reduce[a, a]$ **and**
red-if-equal: $\bigwedge a \ b \ A. a = b : A \implies Reduce[a, b]$ **and**
trans-red: $\bigwedge a \ b \ c \ A. \llbracket a = b : A; Reduce[b, c] \rrbracket \implies a = c : A$ **and**

— Reflexivity

refl-type: $\bigwedge A. A \text{ type} \implies A = A$ **and**
refl-elem: $\bigwedge a \ A. a : A \implies a = a : A$ **and**

— Symmetry

sym-type: $\bigwedge A \ B. A = B \implies B = A$ **and**
sym-elem: $\bigwedge a \ b \ A. a = b : A \implies b = a : A$ **and**

— Transitivity

trans-type: $\bigwedge A \ B \ C. \llbracket A = B; B = C \rrbracket \implies A = C$ **and**
trans-elem: $\bigwedge a \ b \ c \ A. \llbracket a = b : A; b = c : A \rrbracket \implies a = c : A$ **and**

equal-types: $\bigwedge a A B. \llbracket a : A; A = B \rrbracket \Longrightarrow a : B$ **and**
equal-typesL: $\bigwedge a b A B. \llbracket a = b : A; A = B \rrbracket \Longrightarrow a = b : B$ **and**

— Substitution

subst-type: $\bigwedge a A B. \llbracket a : A; \bigwedge z. z:A \Longrightarrow B(z) \text{ type} \rrbracket \Longrightarrow B(a) \text{ type}$ **and**
subst-typeL: $\bigwedge a c A B D. \llbracket a = c : A; \bigwedge z. z:A \Longrightarrow B(z) = D(z) \rrbracket \Longrightarrow B(a) = D(c)$ **and**

subst-elim: $\bigwedge a b A B. \llbracket a : A; \bigwedge z. z:A \Longrightarrow b(z):B(z) \rrbracket \Longrightarrow b(a):B(a)$ **and**
subst-elimL:
 $\bigwedge a b c d A B. \llbracket a = c : A; \bigwedge z. z:A \Longrightarrow b(z)=d(z) : B(z) \rrbracket \Longrightarrow b(a)=d(c) : B(a)$
and

— The type N – natural numbers

NF: N type **and**
NI0: $0 : N$ **and**
NI-succ: $\bigwedge a. a : N \Longrightarrow \text{succ}(a) : N$ **and**
NI-succL: $\bigwedge a b. a = b : N \Longrightarrow \text{succ}(a) = \text{succ}(b) : N$ **and**

NE:
 $\bigwedge p a b C. \llbracket p : N; a : C(0); \bigwedge u v. \llbracket u : N; v : C(u) \rrbracket \Longrightarrow b(u,v) : C(\text{succ}(u)) \rrbracket$
 $\Longrightarrow \text{rec}(p, a, \lambda u v. b(u,v)) : C(p)$ **and**

NEL:
 $\bigwedge p q a b c d C. \llbracket p = q : N; a = c : C(0);$
 $\bigwedge u v. \llbracket u : N; v : C(u) \rrbracket \Longrightarrow b(u,v) = d(u,v) : C(\text{succ}(u)) \rrbracket$
 $\Longrightarrow \text{rec}(p, a, \lambda u v. b(u,v)) = \text{rec}(q, c, d) : C(p)$ **and**

NC0:
 $\bigwedge a b C. \llbracket a : C(0); \bigwedge u v. \llbracket u : N; v : C(u) \rrbracket \Longrightarrow b(u,v) : C(\text{succ}(u)) \rrbracket$
 $\Longrightarrow \text{rec}(0, a, \lambda u v. b(u,v)) = a : C(0)$ **and**

NC-succ:
 $\bigwedge p a b C. \llbracket p : N; a : C(0); \bigwedge u v. \llbracket u : N; v : C(u) \rrbracket \Longrightarrow b(u,v) : C(\text{succ}(u)) \rrbracket \Longrightarrow$
 $\text{rec}(\text{succ}(p), a, \lambda u v. b(u,v)) = b(p, \text{rec}(p, a, \lambda u v. b(u,v))) : C(\text{succ}(p))$ **and**

— The fourth Peano axiom. See page 91 of Martin-Löf's book.

zero-ne-succ: $\bigwedge a. \llbracket a : N; 0 = \text{succ}(a) : N \rrbracket \Longrightarrow 0 : F$ **and**

— The Product of a family of types

ProdF: $\bigwedge A B. \llbracket A \text{ type}; \bigwedge x. x:A \Longrightarrow B(x) \text{ type} \rrbracket \Longrightarrow \prod x:A. B(x) \text{ type}$ **and**

ProdFL:
 $\bigwedge A B C D. \llbracket A = C; \bigwedge x. x:A \Longrightarrow B(x) = D(x) \rrbracket \Longrightarrow \prod x:A. B(x) = \prod x:C.$

$D(x)$ **and**

ProdI:

$\bigwedge b \ A \ B. \llbracket A \text{ type}; \bigwedge x. x:A \implies b(x):B(x) \rrbracket \implies \lambda x. b(x) : \prod x:A. B(x)$ **and**

ProdIL: $\bigwedge b \ c \ A \ B. \llbracket A \text{ type}; \bigwedge x. x:A \implies b(x) = c(x) : B(x) \rrbracket \implies$

$\lambda x. b(x) = \lambda x. c(x) : \prod x:A. B(x)$ **and**

ProdE: $\bigwedge p \ a \ A \ B. \llbracket p : \prod x:A. B(x); a : A \rrbracket \implies p'a : B(a)$ **and**

ProdEL: $\bigwedge p \ q \ a \ b \ A \ B. \llbracket p = q : \prod x:A. B(x); a = b : A \rrbracket \implies p'a = q'b : B(a)$
and

ProdC: $\bigwedge a \ b \ A \ B. \llbracket a : A; \bigwedge x. x:A \implies b(x) : B(x) \rrbracket \implies (\lambda x. b(x))' a = b(a) : B(a)$ **and**

ProdC2: $\bigwedge p \ A \ B. p : \prod x:A. B(x) \implies (\lambda x. p'x) = p : \prod x:A. B(x)$ **and**

— The Sum of a family of types

SumF: $\bigwedge A \ B. \llbracket A \text{ type}; \bigwedge x. x:A \implies B(x) \text{ type} \rrbracket \implies \sum x:A. B(x) \text{ type}$ **and**

SumFL: $\bigwedge A \ B \ C \ D. \llbracket A = C; \bigwedge x. x:A \implies B(x) = D(x) \rrbracket \implies \sum x:A. B(x) = \sum x:C. D(x)$ **and**

SumI: $\bigwedge a \ b \ A \ B. \llbracket a : A; b : B(a) \rrbracket \implies \langle a, b \rangle : \sum x:A. B(x)$ **and**

SumIL: $\bigwedge a \ b \ c \ d \ A \ B. \llbracket a = c : A; b = d : B(a) \rrbracket \implies \langle a, b \rangle = \langle c, d \rangle : \sum x:A. B(x)$ **and**

SumE: $\bigwedge p \ c \ A \ B \ C. \llbracket p : \sum x:A. B(x); \bigwedge x \ y. \llbracket x:A; y:B(x) \rrbracket \implies c(x,y) : C(\langle x, y \rangle) \rrbracket \implies \text{split}(p, \lambda x \ y. c(x,y)) : C(p)$ **and**

SumEL: $\bigwedge p \ q \ c \ d \ A \ B \ C. \llbracket p = q : \sum x:A. B(x);$

$\bigwedge x \ y. \llbracket x:A; y:B(x) \rrbracket \implies c(x,y) = d(x,y) : C(\langle x, y \rangle) \rrbracket$

$\implies \text{split}(p, \lambda x \ y. c(x,y)) = \text{split}(q, \lambda x \ y. d(x,y)) : C(p)$ **and**

SumC: $\bigwedge a \ b \ c \ A \ B \ C. \llbracket a : A; b : B(a); \bigwedge x \ y. \llbracket x:A; y:B(x) \rrbracket \implies c(x,y) : C(\langle x, y \rangle) \rrbracket$

$\implies \text{split}(\langle a, b \rangle, \lambda x \ y. c(x,y)) = c(a,b) : C(\langle a, b \rangle)$ **and**

fst-def: $\bigwedge a. \text{fst}(a) \equiv \text{split}(a, \lambda x \ y. x)$ **and**

snd-def: $\bigwedge a. \text{snd}(a) \equiv \text{split}(a, \lambda x \ y. y)$ **and**

— The sum of two types

PlusF: $\bigwedge A \ B. \llbracket A \text{ type}; B \text{ type} \rrbracket \implies A+B \text{ type}$ **and**

PlusFL: $\bigwedge A \ B \ C \ D. \llbracket A = C; B = D \rrbracket \implies A+B = C+D$ **and**

PlusI-inl: $\bigwedge a \ A \ B. \llbracket a : A; B \text{ type} \rrbracket \implies \text{inl}(a) : A+B$ **and**

PlusI-inlL: $\bigwedge a \ c \ A \ B. \llbracket a = c : A; B \text{ type} \rrbracket \implies \text{inl}(a) = \text{inl}(c) : A+B$ **and**

PlusI-inr: $\bigwedge b A B. \llbracket A \text{ type}; b : B \rrbracket \implies \text{inr}(b) : A+B$ **and**
PlusI-inrL: $\bigwedge b d A B. \llbracket A \text{ type}; b = d : B \rrbracket \implies \text{inr}(b) = \text{inr}(d) : A+B$ **and**

PlusE:

$\bigwedge p c d A B C. \llbracket p : A+B;$
 $\bigwedge x. x:A \implies c(x) : C(\text{inl}(x));$
 $\bigwedge y. y:B \implies d(y) : C(\text{inr}(y)) \rrbracket \implies \text{when}(p, \lambda x. c(x), \lambda y. d(y)) : C(p)$ **and**

PlusEL:

$\bigwedge p q c d e f A B C. \llbracket p = q : A+B;$
 $\bigwedge x. x : A \implies c(x) = e(x) : C(\text{inl}(x));$
 $\bigwedge y. y : B \implies d(y) = f(y) : C(\text{inr}(y)) \rrbracket$
 $\implies \text{when}(p, \lambda x. c(x), \lambda y. d(y)) = \text{when}(q, \lambda x. e(x), \lambda y. f(y)) : C(p)$ **and**

PlusC-inl:

$\bigwedge a c d A B C. \llbracket a : A;$
 $\bigwedge x. x:A \implies c(x) : C(\text{inl}(x));$
 $\bigwedge y. y:B \implies d(y) : C(\text{inr}(y)) \rrbracket$
 $\implies \text{when}(\text{inl}(a), \lambda x. c(x), \lambda y. d(y)) = c(a) : C(\text{inl}(a))$ **and**

PlusC-inr:

$\bigwedge b c d A B C. \llbracket b : B;$
 $\bigwedge x. x:A \implies c(x) : C(\text{inl}(x));$
 $\bigwedge y. y:B \implies d(y) : C(\text{inr}(y)) \rrbracket$
 $\implies \text{when}(\text{inr}(b), \lambda x. c(x), \lambda y. d(y)) = d(b) : C(\text{inr}(b))$ **and**

— The type *Eq*

EqF: $\bigwedge a b A. \llbracket A \text{ type}; a : A; b : A \rrbracket \implies \text{Eq}(A,a,b) \text{ type}$ **and**

EqFL: $\bigwedge a b c d A B. \llbracket A = B; a = c : A; b = d : A \rrbracket \implies \text{Eq}(A,a,b) = \text{Eq}(B,c,d)$ **and**

EqI: $\bigwedge a b A. a = b : A \implies \text{eq} : \text{Eq}(A,a,b)$ **and**

EqE: $\bigwedge p a b A. p : \text{Eq}(A,a,b) \implies a = b : A$ **and**

— By equality of types, can prove $C(p)$ from $C(\text{eq})$, an elimination rule

EqC: $\bigwedge p a b A. p : \text{Eq}(A,a,b) \implies p = \text{eq} : \text{Eq}(A,a,b)$ **and**

— The type *F*

FF: $F \text{ type}$ **and**

FE: $\bigwedge p C. \llbracket p : F; C \text{ type} \rrbracket \implies \text{contr}(p) : C$ **and**

FEL: $\bigwedge p q C. \llbracket p = q : F; C \text{ type} \rrbracket \implies \text{contr}(p) = \text{contr}(q) : C$ **and**

— The type *T*

— Martin-Löf's book (page 68) discusses elimination and computation. Elim-

ination can be derived by computation and equality of types, but with an extra premise $C(x)$ type $x:T$. Also computation can be derived from elimination.

TF: T type **and**
TI: $tt : T$ **and**
TE: $\bigwedge p \ c \ C. \llbracket p : T; c : C(tt) \rrbracket \implies c : C(p)$ **and**
TEL: $\bigwedge p \ q \ c \ d \ C. \llbracket p = q : T; c = d : C(tt) \rrbracket \implies c = d : C(p)$ **and**
TC: $\bigwedge p. p : T \implies p = tt : T$

1.1 Tactics and derived rules for Constructive Type Theory

Formation rules.

lemmas *form-rls* = *NF ProdF SumF PlusF EqF FF TF*
and *formL-rls* = *ProdFL SumFL PlusFL EqFL*

Introduction rules. OMITTED:

- *EqI*, because its premise is an *equelem*, not an *elem*.

lemmas *intr-rls* = *NI0 NI-succ ProdI SumI PlusI-inl PlusI-inr TI*
and *intrL-rls* = *NI-succL ProdIL SumIL PlusI-inlL PlusI-inrL*

Elimination rules. OMITTED:

- *EqE*, because its conclusion is an *equelem*, not an *elem*
- *TE*, because it does not involve a constructor.

lemmas *elim-rls* = *NE ProdE SumE PlusE FE*
and *elimL-rls* = *NEL ProdEL SumEL PlusEL FEL*

OMITTED: *eqC* are *TC* because they make rewriting loop: $p = un = un = \dots$

lemmas *comp-rls* = *NC0 NC-succ ProdC SumC PlusC-inl PlusC-inr*

Rules with conclusion $a:A$, an *elem* judgment.

lemmas *element-rls* = *intr-rls elim-rls*

Definitions are (meta)equality axioms.

lemmas *basic-defs* = *fst-def snd-def*

Compare with standard version: *B* is applied to UNSIMPLIFIED expression!

lemma *SumIL2*: $\llbracket c = a : A; d = b : B(a) \rrbracket \implies \langle c, d \rangle = \langle a, b \rangle : \text{Sum}(A, B)$
by (*rule sym-elem*) (*rule SumIL*; *rule sym-elem*)

lemmas *intrL2-rls* = *NI-succL ProdIL SumIL2 PlusI-inlL PlusI-inrL*

Exploit $p:Prod(A,B)$ to create the assumption $z:B(a)$. A more natural form of product elimination.

```

lemma subst-prodE:
  assumes  $p: Prod(A,B)$ 
    and  $a: A$ 
    and  $\bigwedge z. z: B(a) \implies c(z): C(z)$ 
  shows  $c(p'a): C(p'a)$ 
  by (rule assms ProdE)+

```

1.2 Tactics for type checking

```

ML <
  local

```

```

  fun is-rigid-elem Const-<Elem for a -> = not(is-Var (head-of a))
    | is-rigid-elem Const-<Eelem for a -> = not(is-Var (head-of a))
    | is-rigid-elem Const-<Type for a> = not(is-Var (head-of a))
    | is-rigid-elem - = false

```

```

  in

```

```

  (*Try solving a:A or a=b:A by assumption provided a is rigid!*)
  fun test-assume-tac ctxt = SUBGOAL (fn (prem, i) =>
    if is-rigid-elem (Logic.strip-assums-concl prem)
    then assume-tac ctxt i else no-tac)

```

```

  fun ASSUME ctxt tf i = test-assume-tac ctxt i ORELSE tf i

```

```

  end
>

```

For simplification: type formation and checking, but no equalities between terms.

```

lemmas routine-rls = form-rls formL-rls refl-type element-rls

```

```

ML <

```

```

  fun routine-tac rls ctxt prems =
    ASSUME ctxt (Bires.filt-resolve-from-net-tac ctxt 4 (Bires.build-net (prems @
    rls)));

```

```

  (*Solve all subgoals A type using formation rules. *)
  val form-net = Bires.build-net @ {thms form-rls};
  fun form-tac ctxt =
    REPEAT-FIRST (ASSUME ctxt (Bires.filt-resolve-from-net-tac ctxt 1 form-net));

```

```

  (*Type checking: solve a:A (a rigid, A flexible) by intro and elim rules. *)
  fun typechk-tac ctxt thms =
    let val tac =
      Bires.filt-resolve-from-net-tac ctxt 3

```



```

    (Bires.build-net (thms @ @{thms form-rls} @ @{thms element-rls}))
  in REPEAT-FIRST (ASSUME ctxt tac) end

(*Solve a:A (a flexible, A rigid) by introduction rules.
  Cannot use stringtrees (filt-resolve-tac) since
  goals like ?a:SUM(A,B) have a trivial head-string *)
fun intr-tac ctxt thms =
  let val tac =
    Bires.filt-resolve-from-net-tac ctxt 1
    (Bires.build-net (thms @ @{thms form-rls} @ @{thms intr-rls}))
  in REPEAT-FIRST (ASSUME ctxt tac) end

(*Equality proving: solve a=b:A (where a is rigid) by long rules. *)
fun equal-tac ctxt thms =
  REPEAT-FIRST
    (ASSUME ctxt
      (Bires.filt-resolve-from-net-tac ctxt 3
        (Bires.build-net (thms @ @{thms form-rls element-rls intrL-rls elimL-rls
          refl-elem}))))))
>

method-setup form = <Scan.succeed (fn ctxt => SIMPLE-METHOD (form-tac
  ctxt))>
method-setup typechk = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD
  (typechk-tac ctxt ths))>
method-setup intr = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD
  (intr-tac ctxt ths))>
method-setup equal = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD
  (equal-tac ctxt ths))>

```

1.3 Simplification

To simplify the type in a goal.

```

lemma replace-type:  $\llbracket B = A; a : A \rrbracket \implies a : B$ 
  apply (rule equal-types)
  apply (rule-tac [2] sym-type)
  apply assumption+
  done

```

Simplify the parameter of a unary type operator.

```

lemma subst-eqtyparg:
  assumes 1:  $a=c : A$ 
    and 2:  $\bigwedge z. z:A \implies B(z)$  type
  shows  $B(a) = B(c)$ 
  apply (rule subst-typeL)
  apply (rule-tac [2] refl-type)
  apply (rule 1)
  apply (erule 2)
  done

```

Simplification rules for Constructive Type Theory.

lemmas *reduction-rls* = *comp-rls* [THEN *trans-elem*]

ML <

(*Converts each goal $e : Eq(A,a,b)$ into $a=b:A$ for simplification.

Uses other intro rules to avoid changing flexible goals.*)

val eqintr-net = *Bires.build-net* @{thms *EqI intr-rls*}

fun eqintr-tac *ctxt* =

REPEAT-FIRST (*ASSUME* *ctxt* (*Bires.filt-resolve-from-net-tac* *ctxt* 1 *eqintr-net*))

(** *Tactics that instantiate CTT-rules.*

Vars in the given terms will be incremented!

*The (rtac EqE i) lets them apply to equality judgments. **)*

fun NE-tac *ctxt* *sp* *i* =

TRY (*resolve-tac* *ctxt* @{thms *EqE*} *i*) *THEN*

Rule-Insts.res-inst-tac *ctxt* [(((*p*, 0), *Position.none*), *sp*)] [] @{thm *NE*} *i*

fun SumE-tac *ctxt* *sp* *i* =

TRY (*resolve-tac* *ctxt* @{thms *EqE*} *i*) *THEN*

Rule-Insts.res-inst-tac *ctxt* [(((*p*, 0), *Position.none*), *sp*)] [] @{thm *SumE*} *i*

fun PlusE-tac *ctxt* *sp* *i* =

TRY (*resolve-tac* *ctxt* @{thms *EqE*} *i*) *THEN*

Rule-Insts.res-inst-tac *ctxt* [(((*p*, 0), *Position.none*), *sp*)] [] @{thm *PlusE*} *i*

(** *Predicate logic reasoning, WITH THINNING!! Procedures adapted from NJ.*

**)

(*Finds $f:Prod(A,B)$ and $a:A$ in the assumptions, concludes there is $z:B(a)$ *)

fun add-mp-tac *ctxt* *i* =

resolve-tac *ctxt* @{thms *subst-prodE*} *i* *THEN* *assume-tac* *ctxt* *i* *THEN* *assume-tac* *ctxt* *i*

(*Finds $P \rightarrow Q$ and P in the assumptions, replaces implication by Q *)

fun mp-tac *ctxt* *i* = *eresolve-tac* *ctxt* @{thms *subst-prodE*} *i* *THEN* *assume-tac* *ctxt* *i*

(*safe when regarded as predicate calculus rules*)

val safe-brls = *sort Bires.subgoals-ord*

[(*true*, @{thm *FE*}), (*true*, *asm-rl*),

(*false*, @{thm *ProdI*}), (*true*, @{thm *SumE*}), (*true*, @{thm *PlusE*})]

val unsafe-brls =

[(*false*, @{thm *PlusI-inl*}), (*false*, @{thm *PlusI-inr*}), (*false*, @{thm *SumI*}),

(*true*, @{thm *subst-prodE*})]

(*0 subgoals vs 1 or more*)

val (*safe0-brls*, *safe-p-brls*) =

List.partition Bires.no-subgoals safe-brls

```

fun safestep-tac ctxt thms i =
  form-tac ctxt ORELSE
  resolve-tac ctxt thms i ORELSE
  biresolve-tac ctxt safe0-brls i ORELSE mp-tac ctxt i ORELSE
  DETERM (biresolve-tac ctxt safep-brls i)

fun safe-tac ctxt thms i = DEPTH-SOLVE-1 (safestep-tac ctxt thms i)

fun step-tac ctxt thms = safestep-tac ctxt thms ORELSE' biresolve-tac ctxt un-
safe-brls

(*Fails unless it solves the goal!*)
fun pc-tac ctxt thms = DEPTH-SOLVE-1 o (step-tac ctxt thms)
>

method-setup eqintr = <Scan.succeed (SIMPLE-METHOD o eqintr-tac)>
method-setup NE = <
  Scan.lift Parse.embedded-inner-syntax >> (fn s => fn ctxt => SIMPLE-METHOD'
  (NE-tac ctxt s))
>
method-setup pc = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD'
  (pc-tac ctxt ths))>
method-setup add-mp = <Scan.succeed (SIMPLE-METHOD' o add-mp-tac)>

ML-file <rew.ML>
method-setup rew = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD
  (rew-tac ctxt ths))>
method-setup hyp-rew = <Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD
  (hyp-rew-tac ctxt ths))>

```

1.4 The elimination rules for fst/snd

```

lemma SumE-fst:  $p : \text{Sum}(A,B) \implies \text{fst}(p) : A$ 
  unfolding basic-defs
  apply (erule SumE)
  apply assumption
  done

```

The first premise must be $p:\text{Sum}(A,B)!!$.

```

lemma SumE-snd:
  assumes major:  $p : \text{Sum}(A,B)$ 
  and A type
  and  $\bigwedge x. x:A \implies B(x)$  type
  shows  $\text{snd}(p) : B(\text{fst}(p))$ 
  unfolding basic-defs
  apply (rule major [THEN SumE])
  apply (rule SumC [THEN subst-eqtyparg, THEN replace-type])

```

apply (*typechk assms*)
done

2 The two-element type (booleans and conditionals)

definition *Bool* :: *t*
 where *Bool* $\equiv T + T$

definition *true* :: *i*
 where *true* $\equiv \text{inl}(tt)$

definition *false* :: *i*
 where *false* $\equiv \text{inr}(tt)$

definition *cond* :: $[i, i, i] \Rightarrow i$
 where *cond*(*a, b, c*) $\equiv \text{when}(a, \lambda\cdot. b, \lambda\cdot. c)$

lemmas *bool-defs* = *Bool-def true-def false-def cond-def*

2.1 Derivation of rules for the type *Bool*

Formation rule.

lemma *boolF*: *Bool* type
 unfolding *bool-defs* **by** *typechk*

Introduction rules for *true*, *false*.

lemma *boolI-true*: *true* : *Bool*
 unfolding *bool-defs* **by** *typechk*

lemma *boolI-false*: *false* : *Bool*
 unfolding *bool-defs* **by** *typechk*

Elimination rule: typing of *cond*.

lemma *boolE*: $\llbracket p : \text{Bool}; a : C(\text{true}); b : C(\text{false}) \rrbracket \Longrightarrow \text{cond}(p, a, b) : C(p)$
 unfolding *bool-defs*
 apply (*typechk; erule TE*)
 apply *typechk*
done

lemma *boolEL*: $\llbracket p = q : \text{Bool}; a = c : C(\text{true}); b = d : C(\text{false}) \rrbracket \Longrightarrow \text{cond}(p, a, b) = \text{cond}(q, c, d) : C(p)$
 unfolding *bool-defs*
 apply (*rule PlusEL*)
 apply (*erule asm-rl refl-elem [THEN TEL]*)
done

Computation rules for *true*, *false*.

```

lemma boolC-true:  $\llbracket a : C(\text{true}); b : C(\text{false}) \rrbracket \implies \text{cond}(\text{true}, a, b) = a : C(\text{true})$ 
  unfolding bool-defs
  apply (rule comp-rls)
    apply typechk
    apply (erule-tac [!] TE)
    apply typechk
  done

```

```

lemma boolC-false:  $\llbracket a : C(\text{true}); b : C(\text{false}) \rrbracket \implies \text{cond}(\text{false}, a, b) = b : C(\text{false})$ 
  unfolding bool-defs
  apply (rule comp-rls)
    apply typechk
    apply (erule-tac [!] TE)
    apply typechk
  done

```

3 Elementary arithmetic

3.1 Arithmetic operators and their definitions

```

definition add ::  $[i, i] \Rightarrow i$  (infixr  $\langle \# + \rangle$  65)
  where  $a \# + b \equiv \text{rec}(a, b, \lambda u v. \text{succ}(v))$ 

```

```

definition diff ::  $[i, i] \Rightarrow i$  (infixr  $\langle - \rangle$  65)
  where  $a - b \equiv \text{rec}(b, a, \lambda u v. \text{rec}(v, 0, \lambda x y. x))$ 

```

```

definition absdiff ::  $[i, i] \Rightarrow i$  (infixr  $\langle |-| \rangle$  65)
  where  $a |-| b \equiv (a - b) \# + (b - a)$ 

```

```

definition mult ::  $[i, i] \Rightarrow i$  (infixr  $\langle \# * \rangle$  70)
  where  $a \# * b \equiv \text{rec}(a, 0, \lambda u v. b \# + v)$ 

```

```

definition mod ::  $[i, i] \Rightarrow i$  (infixr  $\langle \text{mod} \rangle$  70)
  where  $a \text{ mod } b \equiv \text{rec}(a, 0, \lambda u v. \text{rec}(\text{succ}(v) |-| b, 0, \lambda x y. \text{succ}(v)))$ 

```

```

definition div ::  $[i, i] \Rightarrow i$  (infixr  $\langle \text{div} \rangle$  70)
  where  $a \text{ div } b \equiv \text{rec}(a, 0, \lambda u v. \text{rec}(\text{succ}(u) \text{ mod } b, \text{succ}(v), \lambda x y. v))$ 

```

```

lemmas arith-defs = add-def diff-def absdiff-def mult-def mod-def div-def

```

3.2 Proofs about elementary arithmetic: addition, multiplication, etc.

3.2.1 Addition

Typing of *add*: short and long versions.

```

lemma add-typing:  $\llbracket a:N; b:N \rrbracket \implies a \# + b : N$ 
  unfolding arith-defs by typechk

```

lemma *add-typingL*: $\llbracket a = c:N; b = d:N \rrbracket \implies a \# + b = c \# + d : N$
unfolding *arith-defs* **by** *equal*

Computation for *add*: 0 and successor cases.

lemma *addC0*: $b:N \implies 0 \# + b = b : N$
unfolding *arith-defs* **by** *rew*

lemma *addC-succ*: $\llbracket a:N; b:N \rrbracket \implies \text{succ}(a) \# + b = \text{succ}(a \# + b) : N$
unfolding *arith-defs* **by** *rew*

3.2.2 Multiplication

Typing of *mult*: short and long versions.

lemma *mult-typing*: $\llbracket a:N; b:N \rrbracket \implies a \# * b : N$
unfolding *arith-defs* **by** (*typechk add-typing*)

lemma *mult-typingL*: $\llbracket a = c:N; b = d:N \rrbracket \implies a \# * b = c \# * d : N$
unfolding *arith-defs* **by** (*equal add-typingL*)

Computation for *mult*: 0 and successor cases.

lemma *multC0*: $b:N \implies 0 \# * b = 0 : N$
unfolding *arith-defs* **by** *rew*

lemma *multC-succ*: $\llbracket a:N; b:N \rrbracket \implies \text{succ}(a) \# * b = b \# + (a \# * b) : N$
unfolding *arith-defs* **by** *rew*

3.2.3 Difference

Typing of difference.

lemma *diff-typing*: $\llbracket a:N; b:N \rrbracket \implies a - b : N$
unfolding *arith-defs* **by** *typechk*

lemma *diff-typingL*: $\llbracket a = c:N; b = d:N \rrbracket \implies a - b = c - d : N$
unfolding *arith-defs* **by** *equal*

Computation for difference: 0 and successor cases.

lemma *diffC0*: $a:N \implies a - 0 = a : N$
unfolding *arith-defs* **by** *rew*

Note: $\text{rec}(a, 0, \lambda z w.z)$ is $\text{pred}(a)$.

lemma *diff-0-eq-0*: $b:N \implies 0 - b = 0 : N$
unfolding *arith-defs*
by (*NE b*) *hyp-rew*

Essential to simplify FIRST!! (Else we get a critical pair) $\text{succ}(a) - \text{succ}(b)$ rewrites to $\text{pred}(\text{succ}(a) - b)$.

lemma *diff-succ-succ*: $\llbracket a:N; b:N \rrbracket \implies \text{succ}(a) - \text{succ}(b) = a - b : N$

```

unfolding arith-defs
apply hyp-rew
apply (NE b)
  apply hyp-rew
done

```

3.3 Simplification

```

lemmas arith-typing-rls = add-typing mult-typing diff-typing
  and arith-congr-rls = add-typingL mult-typingL diff-typingL

```

```

lemmas congr-rls = arith-congr-rls intrL2-rls elimL-rls

```

```

lemmas arithC-rls =
  addC0 addC-succ
  multC0 multC-succ
  diffC0 diff-0-eq-0 diff-succ-succ

```

```

ML <
  structure Arith-simp = TSimpFun(
    val refl = @{thm refl-elem}
    val sym = @{thm sym-elem}
    val trans = @{thm trans-elem}
    val refl-red = @{thm refl-red}
    val trans-red = @{thm trans-red}
    val red-if-equal = @{thm red-if-equal}
    val default-rls = @{thms arithC-rls comp-rls}
    val routine-tac = routine-tac @{thms arith-typing-rls routine-rls}
  )

  fun arith-rew-tac ctxt prems =
    make-rew-tac ctxt (Arith-simp.norm-tac ctxt (@{thms congr-rls}, prems))

  fun hyp-arith-rew-tac ctxt prems =
    make-rew-tac ctxt
      (Arith-simp.cond-norm-tac ctxt (prove-cond-tac ctxt, @{thms congr-rls},
prems))
  >

method-setup arith-rew = <
  Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD (arith-rew-tac ctxt
ths))
  >

method-setup hyp-arith-rew = <
  Attrib.thms >> (fn ths => fn ctxt => SIMPLE-METHOD (hyp-arith-rew-tac
ctxt ths))
  >

```

3.4 Addition

Associative law for addition.

lemma *add-assoc*: $\llbracket a:N; b:N; c:N \rrbracket \implies (a \# + b) \# + c = a \# + (b \# + c) : N$
by (*NE a*) *hyp-arith-rew*

Commutative law for addition. Can be proved using three inductions. Must simplify after first induction! Orientation of rewrites is delicate.

lemma *add-commute*: $\llbracket a:N; b:N \rrbracket \implies a \# + b = b \# + a : N$
apply (*NE a*)
apply *hyp-arith-rew*
apply (*rule sym-elem*)
prefer 2
apply (*NE b*)
prefer 4
apply (*NE b*)
apply *hyp-arith-rew*
done

3.5 Multiplication

Right annihilation in product.

lemma *mult-0-right*: $a:N \implies a \# * 0 = 0 : N$
apply (*NE a*)
apply *hyp-arith-rew*
done

Right successor law for multiplication.

lemma *mult-succ-right*: $\llbracket a:N; b:N \rrbracket \implies a \# * \text{succ}(b) = a \# + (a \# * b) : N$
apply (*NE a*)
apply (*hyp-arith-rew add-assoc [THEN sym-elem]*)
apply (*assumption | rule add-commute mult-typingL add-typingL intrL-rls refl-elem*)
done

Commutative law for multiplication.

lemma *mult-commute*: $\llbracket a:N; b:N \rrbracket \implies a \# * b = b \# * a : N$
apply (*NE a*)
apply (*hyp-arith-rew mult-0-right mult-succ-right*)
done

Addition distributes over multiplication.

lemma *add-mult-distrib*: $\llbracket a:N; b:N; c:N \rrbracket \implies (a \# + b) \# * c = (a \# * c) \# + (b \# * c) : N$
apply (*NE a*)
apply (*hyp-arith-rew add-assoc [THEN sym-elem]*)
done

Associative law for multiplication.


```

lemma mult-assoc:  $\llbracket a:N; b:N; c:N \rrbracket \implies (a \#* b) \#* c = a \#* (b \#* c) : N$ 
  apply (NE a)
    apply (hyp-arith-rew add-mult-distrib)
  done

```

3.6 Difference

Difference on natural numbers, without negative numbers

- $a - b = 0$ iff $a \leq b$
- $a - b = \text{succ}(c)$ iff $a > b$

```

lemma diff-self-eq-0:  $a:N \implies a - a = 0 : N$ 
  apply (NE a)
    apply hyp-arith-rew
  done

```

```

lemma add-0-right:  $\llbracket c : N; 0 : N; c : N \rrbracket \implies c \# + 0 = c : N$ 
  by (rule addC0 [THEN [3] add-commute [THEN trans-elem]])

```

Addition is the inverse of subtraction: if $b \leq x$ then $b \# + (x - b) = x$. An example of induction over a quantified formula (a product). Uses rewriting with a quantified, implicative inductive hypothesis.

```

schematic-goal add-diff-inverse-lemma:
   $b:N \implies ?a : \prod x:N. Eq(N, b-x, 0) \longrightarrow Eq(N, b \# + (x-b), x)$ 
  apply (NE b)
    — strip one "universal quantifier" but not the "implication"
    apply (rule-tac [3] intr-rls)
    — case analysis on  $x$  in  $\text{succ}(u) \leq x \longrightarrow \text{succ}(u) \# + (x - \text{succ}(u)) = x$ 
    prefer 4
    apply (NE x)
      apply assumption
      — Prepare for simplification of types – the antecedent  $\text{succ}(u) \leq x$ 
      apply (rule-tac [2] replace-type)
      apply (rule-tac [1] replace-type)
      apply arith-rew
      — Solves first 0 goal, simplifies others. Two subgoals remain. Both follow by
rewriting, (2) using quantified induction hyp.
      apply intr — strips remaining  $\prod$ s
      apply (hyp-arith-rew add-0-right)
    apply assumption
  done

```

Version of above with premise $b - a = 0$ i.e. $a \geq b$. Using *ProdE* does not work – for $?B(?a)$ is ambiguous. Instead, *add-diff-inverse-lemma* states the desired induction scheme; the use of *THEN* below instantiates Vars in *ProdE* automatically.

```

lemma add-diff-inverse:  $\llbracket a:N; b:N; b - a = 0 : N \rrbracket \implies b \# + (a-b) = a : N$ 
  apply (rule EqE)
  apply (rule add-diff-inverse-lemma [THEN ProdE, THEN ProdE])
  apply (assumption | rule EqI) +
done

```

3.7 Absolute difference

Typing of absolute difference: short and long versions.

```

lemma absdiff-typing:  $\llbracket a:N; b:N \rrbracket \implies a \mid - \mid b : N$ 
  unfolding arith-defs by typechk

```

```

lemma absdiff-typingL:  $\llbracket a = c:N; b = d:N \rrbracket \implies a \mid - \mid b = c \mid - \mid d : N$ 
  unfolding arith-defs by equal

```

```

lemma absdiff-self-eq-0:  $a:N \implies a \mid - \mid a = 0 : N$ 
  unfolding absdiff-def by (arith-rew diff-self-eq-0)

```

```

lemma absdiffC0:  $a:N \implies 0 \mid - \mid a = a : N$ 
  unfolding absdiff-def by hyp-arith-rew

```

```

lemma absdiff-succ-succ:  $\llbracket a:N; b:N \rrbracket \implies \text{succ}(a) \mid - \mid \text{succ}(b) = a \mid - \mid b : N$ 
  unfolding absdiff-def by hyp-arith-rew

```

Note how easy using commutative laws can be? ...not always...

```

lemma absdiff-commute:  $\llbracket a:N; b:N \rrbracket \implies a \mid - \mid b = b \mid - \mid a : N$ 
  unfolding absdiff-def
  by (rule add-commute) (typechk diff-typing)

```

If $a + b = 0$ then $a = 0$. Surprisingly tedious.

```

schematic-goal add-eq0-lemma:  $\llbracket a:N; b:N \rrbracket \implies ?c : \text{Eq}(N, a \# + b, 0) \longrightarrow \text{Eq}(N, a, 0)$ 
  apply (NE a)
  apply (rule-tac [3] replace-type)
  apply arith-rew
  apply intr — strips remaining  $\prod$ s
  apply (rule-tac [2] zero-ne-succ [THEN FE])
  apply (rule-tac [3] EqE [THEN sym-elem])
  apply (typechk add-typing)
done

```

Version of above with the premise $a + b = 0$. Again, resolution instantiates variables in *ProdE*.

```

lemma add-eq0:  $\llbracket a:N; b:N; a \# + b = 0 : N \rrbracket \implies a = 0 : N$ 
  apply (rule EqE)
  apply (rule add-eq0-lemma [THEN ProdE])
  apply (rule-tac [3] EqI)
  apply typechk
done

```

Here is a lemma to infer $a - b = 0$ and $b - a = 0$ from $a \mid - \mid b = 0$, below.

schematic-goal *absdiff-eq0-lem*:

```

 $\llbracket a:N; b:N; a \mid - \mid b = 0 : N \rrbracket \implies ?a : Eq(N, a-b, 0) \times Eq(N, b-a, 0)$ 
unfolding absdiff-def
apply intr
apply eqintr
apply (rule-tac [2] add-eq0)
apply (rule add-eq0)
apply (rule-tac [6] add-commute [THEN trans-elem])
apply (typechk diff-typing)
done

```

If $a \mid - \mid b = 0$ then $a = b$ proof: $a - b = 0$ and $b - a = 0$, so $b = a + (b - a) = a + 0 = a$.

lemma *absdiff-eq0*: $\llbracket a \mid - \mid b = 0 : N; a:N; b:N \rrbracket \implies a = b : N$

```

apply (rule EqE)
apply (rule absdiff-eq0-lem [THEN SumE])
apply eqintr
apply (rule add-diff-inverse [THEN sym-elem, THEN trans-elem])
apply (erule-tac [3] EqE)
apply (hyp-arith-rew add-0-right)
done

```

3.8 Remainder and Quotient

Typing of remainder: short and long versions.

lemma *mod-typing*: $\llbracket a:N; b:N \rrbracket \implies a \bmod b : N$
unfolding *mod-def* **by** (*typechk absdiff-typing*)

lemma *mod-typingL*: $\llbracket a = c:N; b = d:N \rrbracket \implies a \bmod b = c \bmod d : N$
unfolding *mod-def* **by** (*equal absdiff-typingL*)

Computation for *mod*: 0 and successor cases.

lemma *modC0*: $b:N \implies 0 \bmod b = 0 : N$
unfolding *mod-def* **by** (*rew absdiff-typing*)

lemma *modC-succ*: $\llbracket a:N; b:N \rrbracket \implies$
 $\text{succ}(a) \bmod b = \text{rec}(\text{succ}(a \bmod b) \mid - \mid b, 0, \lambda x y. \text{succ}(a \bmod b)) : N$
unfolding *mod-def* **by** (*rew absdiff-typing*)

Typing of quotient: short and long versions.

lemma *div-typing*: $\llbracket a:N; b:N \rrbracket \implies a \div b : N$
unfolding *div-def* **by** (*typechk absdiff-typing mod-typing*)

lemma *div-typingL*: $\llbracket a = c:N; b = d:N \rrbracket \implies a \div b = c \div d : N$
unfolding *div-def* **by** (*equal absdiff-typingL mod-typingL*)

lemmas *div-typing-rls* = *mod-typing div-typing absdiff-typing*

Computation for quotient: 0 and successor cases.

lemma *divC0*: $b:N \implies 0 \text{ div } b = 0 : N$
unfolding *div-def* **by** (*rew mod-typing absdiff-typing*)

lemma *divC-succ*: $\llbracket a:N; b:N \rrbracket \implies$
 $\text{succ}(a) \text{ div } b = \text{rec}(\text{succ}(a) \text{ mod } b, \text{succ}(a \text{ div } b), \lambda x y. a \text{ div } b) : N$
unfolding *div-def* **by** (*rew mod-typing*)

Version of above with same condition as the *mod* one.

lemma *divC-succ2*: $\llbracket a:N; b:N \rrbracket \implies$
 $\text{succ}(a) \text{ div } b = \text{rec}(\text{succ}(a \text{ mod } b) \mid\mid b, \text{succ}(a \text{ div } b), \lambda x y. a \text{ div } b) : N$
apply (*rule divC-succ [THEN trans-elem]*)
apply (*rew div-typing-rls modC-succ*)
apply (*NE succ (a mod b) \mid\mid b*)
apply (*rew mod-typing div-typing absdiff-typing*)
done

For case analysis on whether a number is 0 or a successor.

lemma *iszero-decidable*: $a:N \implies \text{rec}(a, \text{inl}(eq), \lambda ka kb. \text{inr}(<ka, eq>)) :$
 $Eq(N, a, 0) + (\sum x:N. Eq(N, a, \text{succ}(x)))$
apply (*NE a*)
apply (*rule-tac [3] PlusI-inr*)
apply (*rule-tac [2] PlusI-inl*)
apply *eqintr*
apply *equal*
done

Main Result. Holds when b is 0 since $a \text{ mod } 0 = a$ and $a \text{ div } 0 = 0$.

lemma *mod-div-equality*: $\llbracket a:N; b:N \rrbracket \implies a \text{ mod } b \# + (a \text{ div } b) \# * b = a : N$
apply (*NE a*)
apply (*arith-rew div-typing-rls modC0 modC-succ divC0 divC-succ2*)
apply (*rule EqE*)
— case analysis on $\text{succ}(u \text{ mod } b) \mid\mid b$
apply (*rule-tac a1 = succ (u mod b) \mid\mid b in iszero-decidable [THEN PlusE]*)
apply (*erule-tac [3] SumE*)
apply (*hyp-arith-rew div-typing-rls modC0 modC-succ divC0 divC-succ2*)
— Replace one occurrence of b by $\text{succ}(u \text{ mod } b)$. Clumsy!
apply (*rule add-typingL [THEN trans-elem]*)
apply (*erule EqE [THEN absdiff-eq0, THEN sym-elem]*)
apply (*rule-tac [3] refl-elem*)
apply (*hyp-arith-rew div-typing-rls*)
done

end

4 Easy examples: type checking and type deduction

```
theory Typechecking
imports ../CTT
begin
```

4.1 Single-step proofs: verifying that a type is well-formed

```
schematic-goal ?A type
  by (rule form-rls)
```

```
schematic-goal ?A type
  apply (rule form-rls)
  back
  apply (rule form-rls)
  apply (rule form-rls)
  done
```

```
schematic-goal  $\prod z: ?A . N + ?B(z)$  type
  apply (rule form-rls)
  apply (rule form-rls)
  apply (rule form-rls)
  apply (rule form-rls)
  apply (rule form-rls)
  done
```

4.2 Multi-step proofs: Type inference

```
lemma  $\prod w: N . N + N$  type
  by form
```

```
schematic-goal  $<0, succ(0)> : ?A$ 
  apply intr done
```

```
schematic-goal  $\prod w: N . Eq(?A, w, w)$  type
  apply typechk done
```

```
schematic-goal  $\prod x: N . \prod y: N . Eq(?A, x, y)$  type
  apply typechk done
```

typechecking an application of fst

```
schematic-goal  $(\lambda u. split(u, \lambda v w. v)) ' <0, succ(0)> : ?A$ 
  apply typechk done
```

typechecking the predecessor function

```
schematic-goal  $\lambda n. rec(n, 0, \lambda x y. x) : ?A$ 
  apply typechk done
```

typechecking the addition function

```
schematic-goal  $\lambda n. \lambda m. \text{rec}(n, m, \lambda x y. \text{succ}(y)) : ?A$   
  apply typechk done
```

Proofs involving arbitrary types. For concreteness, every type variable left over is forced to be N

```
method-setup  $N =$   
   $\langle \text{Scan.succeed } (\text{fn } \text{ctxt} \Rightarrow \text{SIMPLE-METHOD } (\text{TRYALL } (\text{resolve-tac } \text{ctxt } @\{\text{thms } NF\}))) \rangle$ 
```

```
schematic-goal  $\lambda w. \langle w, w \rangle : ?A$   
  apply typechk  
  apply  $N$   
  done
```

```
schematic-goal  $\lambda x. \lambda y. x : ?A$   
  apply typechk  
  apply  $N$   
  done
```

typechecking fst (as a function object)

```
schematic-goal  $\lambda i. \text{split}(i, \lambda j k. j) : ?A$   
  apply typechk  
  apply  $N$   
  done
```

end

5 Examples with elimination rules

```
theory Elimination  
imports ../CTT  
begin
```

This finds the functions fst and snd!

```
schematic-goal [folded basic-defs]:  $A \text{ type} \Longrightarrow ?a : (A \times A) \longrightarrow A$   
  apply pc  
  done
```

```
schematic-goal [folded basic-defs]:  $A \text{ type} \Longrightarrow ?a : (A \times A) \longrightarrow A$   
  apply pc  
  back  
  done
```

Double negation of the Excluded Middle

```
schematic-goal  $A \text{ type} \Longrightarrow ?a : ((A + (A \longrightarrow F)) \longrightarrow F) \longrightarrow F$   
  apply intr
```

```

apply (rule ProdE)
apply assumption
apply pc
done

```

Experiment: the proof above in Isar

```

lemma
  assumes  $A$  type shows  $(\lambda f. f \text{ ' } inr(\lambda y. f \text{ ' } inl(y))) : ((A + (A \longrightarrow F)) \longrightarrow F)$ 
   $\longrightarrow F$ 
proof intr
  fix  $f$ 
  assume  $f: f : A + (A \longrightarrow F) \longrightarrow F$ 
  with assms have  $inr(\lambda y. f \text{ ' } inl(y)) : A + (A \longrightarrow F)$ 
  by pc
  then show  $f \text{ ' } inr(\lambda y. f \text{ ' } inl(y)) : F$ 
  by (rule ProdE [OF  $f$ ])
qed (rule assms)+

```

```

schematic-goal  $\llbracket A \text{ type}; B \text{ type} \rrbracket \Longrightarrow ?a : (A \times B) \longrightarrow (B \times A)$ 
apply pc
done

```

Binary sums and products

```

schematic-goal  $\llbracket A \text{ type}; B \text{ type}; C \text{ type} \rrbracket \Longrightarrow ?a : (A + B \longrightarrow C) \longrightarrow (A \longrightarrow C)$ 
 $\times (B \longrightarrow C)$ 
apply pc
done

```

```

schematic-goal  $\llbracket A \text{ type}; B \text{ type}; C \text{ type} \rrbracket \Longrightarrow ?a : A \times (B + C) \longrightarrow (A \times B +$ 
 $A \times C)$ 
by pc

```

```

schematic-goal
  assumes  $A$  type
  and  $\bigwedge x. x:A \Longrightarrow B(x)$  type
  and  $\bigwedge x. x:A \Longrightarrow C(x)$  type
  shows  $?a : (\sum x:A. B(x) + C(x)) \longrightarrow (\sum x:A. B(x)) + (\sum x:A. C(x))$ 
  apply (pc assms)
done

```

Construction of the currying functional

```

schematic-goal  $\llbracket A \text{ type}; B \text{ type}; C \text{ type} \rrbracket \Longrightarrow ?a : (A \times B \longrightarrow C) \longrightarrow (A \longrightarrow (B$ 
 $\longrightarrow C))$ 
apply pc
done

```

```

schematic-goal
  assumes  $A$  type
    and  $\bigwedge x. x:A \implies B(x)$  type
    and  $\bigwedge z. z: (\sum x:A. B(x)) \implies C(z)$  type
  shows  $?a : \prod f: (\prod z : (\sum x:A . B(x)) . C(z)).$ 
     $(\prod x:A . \prod y:B(x) . C(<x,y>))$ 
  apply (pc assms)
  done

```

Martin-Löf (1984), page 48: axiom of sum-elimination (uncurry)

```

schematic-goal  $\llbracket A \text{ type}; B \text{ type}; C \text{ type} \rrbracket \Longrightarrow ?a : (A \longrightarrow (B \longrightarrow C)) \longrightarrow (A \times B \longrightarrow C)$ 
  apply pc
  done

```

```

schematic-goal
  assumes  $A$  type
    and  $\bigwedge x. x:A \implies B(x)$  type
    and  $\bigwedge z. z: (\sum x:A . B(x)) \implies C(z)$  type
  shows  $?a : (\prod x:A . \prod y:B(x) . C(\langle x, y \rangle))$ 
     $\longrightarrow (\prod z : (\sum x:A . B(x)) . C(z))$ 
  apply (pc assms)
  done

```

Function application

schematic-goal $\llbracket A \text{ type}; B \text{ type} \rrbracket \Longrightarrow ?a : ((A \longrightarrow B) \times A) \longrightarrow B$
apply pc
done

Basic test of quantifier reasoning

schematic-goal
 assumes A type
 and B type
 and $\bigwedge x y. \llbracket x:A; y:B \rrbracket \Longrightarrow C(x,y)$ type
shows

$$\begin{aligned} ?a : & \quad (\sum y:B . \prod x:A . C(x,y)) \\ & \longrightarrow (\prod x:A . \sum y:B . C(x,y)) \end{aligned}$$

apply (pc *assms*)
done

Martin-Löf (1984) pages 36-7: the combinator S

$$\begin{array}{l} \text{schema-goal} \\ \text{assumes } A \text{ type} \\ \quad \text{and } \bigwedge x. x:A \implies B(x) \text{ type} \\ \quad \text{and } \bigwedge x y. \llbracket x:A; y:B(x) \rrbracket \implies C(x,y) \text{ type} \\ \text{shows } ?a : \quad (\prod x:A. \prod y:B(x). C(x,y)) \\ \quad \longrightarrow (\prod f: (\prod x:A. B(x)). \prod x:A. C(x, f'x)) \end{array}$$

apply (*pc assms*)
done

Martin-Löf (1984) page 58: the axiom of disjunction elimination

schematic-goal
assumes A *type*
and B *type*
and $\bigwedge z. z: A+B \implies C(z)$ *type*
shows $?a : (\prod x:A. C(\text{inl}(x))) \longrightarrow (\prod y:B. C(\text{inr}(y)))$
 $\longrightarrow (\prod z: A+B. C(z))$
apply (*pc assms*)
done

schematic-goal [*folded basic-defs*]:
 $\llbracket A \text{ type}; B \text{ type}; C \text{ type} \rrbracket \implies ?a : (A \longrightarrow B \times C) \longrightarrow (A \longrightarrow B) \times (A \longrightarrow C)$
apply *pc*
done

AXIOM OF CHOICE! Delicate use of elimination rules

schematic-goal
assumes A *type*
and $\bigwedge x. x:A \implies B(x)$ *type*
and $\bigwedge x y. \llbracket x:A; y:B(x) \rrbracket \implies C(x,y)$ *type*
shows $?a : (\prod x:A. \sum y:B(x). C(x,y)) \longrightarrow (\sum f: (\prod x:A. B(x)). \prod x:A. C(x, f'x))$
apply (*intr assms*)
prefer 2 **apply** *add-mp*
prefer 2 **apply** *add-mp*
apply (*erule SumE-fst*)
apply (*rule replace-type*)
apply (*rule subst-eqtyparg*)
apply (*rule comp-rls*)
apply (*rule-tac* [4] *SumE-snd*)
apply (*typechk SumE-fst assms*)
done

A structured proof of AC

lemma *Axiom-of-Choice*:
assumes A *type*
and $\bigwedge x. x:A \implies B(x)$ *type*
and $\bigwedge x y. \llbracket x:A; y:B(x) \rrbracket \implies C(x,y)$ *type*
shows $(\lambda f. \langle \lambda x. \text{fst}(f'x), \lambda x. \text{snd}(f'x) \rangle)$
 $: (\prod x:A. \sum y:B(x). C(x,y)) \longrightarrow (\sum f: (\prod x:A. B(x)). \prod x:A. C(x, f'x))$
proof (*intr assms*)
fix f a
assume $f : f : \prod x:A. \text{Sum}(B(x), C(x))$ **and** $a : A$
then have $fa : f'a : \text{Sum}(B(a), C(a))$
by (*rule ProdE*)

```

then show  $\text{fst}(f \text{ ' } a) : B(a)$ 
  by (rule SumE-fst)
have  $\text{snd}(f \text{ ' } a) : C(a, \text{fst}(f \text{ ' } a))$ 
  by (rule SumE-snd [OF fa]) (typechk SumE-fst assms ⟨a : A⟩)
moreover have  $(\lambda x. \text{fst}(f \text{ ' } x)) \text{ ' } a = \text{fst}(f \text{ ' } a) : B(a)$ 
  by (rule ProdC [OF ⟨a : A⟩]) (typechk SumE-fst f)
ultimately show  $\text{snd}(f \text{ ' } a) : C(a, (\lambda x. \text{fst}(f \text{ ' } x)) \text{ ' } a)$ 
  by (intro replace-type [OF subst-eqtyparg]) (typechk SumE-fst assms ⟨a : A⟩)
qed

```

Axiom of choice. Proof without fst, snd. Harder still!

schematic-goal [folded basic-defs]:

```

assumes A type
  and  $\bigwedge x. x:A \implies B(x)$  type
  and  $\bigwedge x y. \llbracket x:A; y:B(x) \rrbracket \implies C(x,y)$  type
shows ?a :  $(\prod x:A. \sum y:B(x). C(x,y)) \longrightarrow (\sum f: (\prod x:A. B(x)). \prod x:A. C(x, f \text{ ' } x))$ 
apply (intr assms)

apply (rule ProdE [THEN SumE])
  apply assumption
  apply assumption
  apply assumption
apply (rule replace-type)
apply (rule subst-eqtyparg)
  apply (rule comp-rls)
  apply (erule-tac [4] ProdE [THEN SumE])
  apply (typechk assms)
apply (rule replace-type)
  apply (rule subst-eqtyparg)
  apply (rule comp-rls)
  apply (typechk assms)
apply assumption
done

```

Example of sequent-style deduction

schematic-goal

```

assumes A type
  and B type
  and  $\bigwedge z. z:A \times B \implies C(z)$  type
shows ?a :  $(\sum z:A \times B. C(z)) \longrightarrow (\sum u:A. \sum v:B. C(\langle u,v \rangle))$ 
apply (rule intr-rls)
  apply (tactic ⟨biresolve-tac context safe-brls 2⟩)

  apply (rule-tac [2] a = y in ProdE)
  apply (typechk assms)
apply (rule SumE, assumption)
apply intr
  defer 1

```

```

    apply assumption+
    apply (typechk asms)
done

```

end

6 Equality reasoning by rewriting

```

theory Equality
  imports ../CTT
begin

```

```

lemma split-eq:  $p : \text{Sum}(A,B) \implies \text{split}(p, \text{pair}) = p : \text{Sum}(A,B)$ 
  apply (rule EqE)
  apply (rule elim-rls, assumption)
  apply rew
done

```

```

lemma when-eq:  $\llbracket A \text{ type}; B \text{ type}; p : A+B \rrbracket \implies \text{when}(p, \text{inl}, \text{inr}) = p : A + B$ 
  apply (rule EqE)
  apply (rule elim-rls, assumption)
  apply rew
done

```

in the "rec" formulation of addition, $0 + n = n$

```

lemma p:N  $\implies \text{rec}(p, 0, \lambda y z. \text{succ}(y)) = p : N$ 
  apply (rule EqE)
  apply (rule elim-rls, assumption)
  apply rew
done

```

the harder version, $n + 0 = n$: recursive, uses induction hypothesis

```

lemma p:N  $\implies \text{rec}(p, 0, \lambda y z. \text{succ}(z)) = p : N$ 
  apply (rule EqE)
  apply (rule elim-rls, assumption)
  apply hyp-rew
done

```

Associativity of addition

```

lemma  $\llbracket a:N; b:N; c:N \rrbracket$ 
 $\implies \text{rec}(\text{rec}(a, b, \lambda x y. \text{succ}(y)), c, \lambda x y. \text{succ}(y)) =$ 
 $\text{rec}(a, \text{rec}(b, c, \lambda x y. \text{succ}(y)), \lambda x y. \text{succ}(y)) : N$ 
  apply (NE a)
  apply hyp-rew
done

```

Martin-Löf (1984) page 62: pairing is surjective

```

lemma  $p : \text{Sum}(A,B) \implies \langle \text{split}(p, \lambda x y. x), \text{split}(p, \lambda x y. y) \rangle = p : \text{Sum}(A,B)$ 

```

```

apply (rule EqE)
apply (rule elim-rls, assumption)
apply (tactic ‹DEPTH-SOLVE-1 (rew-tac context [])›)
done

lemma  $\llbracket a : A; b : B \rrbracket \implies (\lambda u. \text{split}(u, \lambda v w. \langle w, v \rangle)) \text{ ‹} \langle a, b \rangle = \langle b, a \rangle : \sum x:B. A$ 
by rew

a contrived, complicated simplication, requires sum-elimination also
lemma  $(\lambda f. \lambda x. f'(f'x)) \text{ ‹} (\lambda u. \text{split}(u, \lambda v w. \langle w, v \rangle)) =$ 
 $\lambda x. x : \prod x: (\sum y:N. N). (\sum y:N. N)$ 
apply (rule reduction-rls)
apply (rule-tac [3] intrL-rls)
apply (rule-tac [4] EqE)
apply (erule-tac [4] SumE)

apply rew
done

end

```

7 Synthesis examples, using a crude form of narrowing

```

theory Synthesis
imports ../CTT
begin

discovery of predecessor function

schematic-goal  $?a : \sum \text{pred}:?A . \text{Eq}(N, \text{pred}'0, 0) \times (\prod n:N. \text{Eq}(N, \text{pred} \text{ ‹} \text{succ}(n), n))$ 
apply intr
apply eqintr
apply (rule-tac [3] reduction-rls)
apply (rule-tac [5] comp-rls)
apply rew
done

the function fst as an element of a function type

schematic-goal [folded basic-defs]:
 $A \text{ type} \implies ?a: \sum f:?B . \prod i:A. \prod j:A. \text{Eq}(A, f \text{ ‹} \langle i, j \rangle, i)$ 
apply intr
apply eqintr
apply (rule-tac [2] reduction-rls)
apply (rule-tac [4] comp-rls)
apply typechk

```

now put in A everywhere

```
  apply assumption+
done
```

An interesting use of the eliminator, when

```
schematic-goal ?a :  $\prod i:N. Eq(?A, ?b(inl(i)), <0, i>)$ 
  ×  $Eq(?A, ?b(inr(i)), <succ(0), i>)$ 
  apply intr
  apply eqintr
  apply (rule comp-rls)
  apply rew
done
```

```
schematic-goal ?a :  $\prod i:N. Eq(?A(i), ?b(inl(i)), <0, i>)$ 
  ×  $Eq(?A(i), ?b(inr(i)), <succ(0), i>)$ 
oops
```

A tricky combination of when and split

```
schematic-goal [folded basic-defs]:
  ?a :  $\prod i:N. \prod j:N. Eq(?A, ?b(inl(<i,j>)), i)$ 
    ×  $Eq(?A, ?b(inr(<i,j>)), j)$ 
  apply intr
  apply eqintr
  apply (rule PlusC-inl [THEN trans-elem])
  apply (rule-tac [4] comp-rls)
  apply (rule-tac [7] reduction-rls)
  apply (rule-tac [10] comp-rls)
  apply typechk
done
```

```
schematic-goal ?a :  $\prod i:N. \prod j:N. Eq(?A(i,j), ?b(inl(<i,j>)), i)$ 
  ×  $Eq(?A(i,j), ?b(inr(<i,j>)), j)$ 
oops
```

```
schematic-goal ?a :  $\prod i:N. \prod j:N. Eq(N, ?b(inl(<i,j>)), i)$ 
  ×  $Eq(N, ?b(inr(<i,j>)), j)$ 
oops
```

Deriving the addition operator

```
schematic-goal [folded arith-defs]:
  ?c :  $\prod n:N. Eq(N, ?f(0,n), n)$ 
    ×  $(\prod m:N. Eq(N, ?f(succ(m), n), succ(?f(m,n))))$ 
  apply intr
  apply eqintr
  apply (rule comp-rls)
```

```

    apply rew
done

```

The addition function – using explicit lambdas

```

schematic-goal [folded arith-defs]:
  ?c :  $\sum plus : ?A$  .
     $\prod x:N. Eq(N, plus\ '0\ x, x)$ 
       $\times (\prod y:N. Eq(N, plus\ succ(y)\ x, succ(plus\ y\ x)))$ 
  apply intr
  apply eqintr
  apply (tactic resolve-tac context [TSimp.split-eqn] 3)
  apply (tactic SELECT-GOAL (rew-tac context []) 4)
    apply (tactic resolve-tac context [TSimp.split-eqn] 3)
    apply (tactic SELECT-GOAL (rew-tac context []) 4)
      apply (rule-tac [3]  $p = y$  in NC-succ)

    apply rew
done
end

```